PHYSICS OF FLUIDS **24**, 054107 (2012)

Secularly growing oscillations in a stratified rotating fluid

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(Received 18 January 2012; accepted 30 April 2012; published online 30 May 2012)

A simple exact solution of the Boussinesq equations of motion, thermal energy, and mass conservation is obtained for an oscillatory flow regime in a stably stratified rotating fluid. The flow is unbounded and characterized by velocity gradients that vary with time but are spatially uniform—the basic state of Craik-Criminale flows. The solution describes an oscillatory convergent-divergent flow in which the linear (normal) strain rates are periodic. However, two of the shear strain rates are forced by the linear strain rates, and their amplitudes grow linearly with time. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4722351]

I. INTRODUCTION

Idealized flow models in which the velocity components vary linearly with one or more of the spatial coordinates have been applied in many contexts. Such models have led to wave and vortex solutions of the nonlinear shallow-water equations,[1–4] the multi-layer reduced shallow-water equations,[5,6] the Euler equations,[7] and the Boussinesq equations for a stratified fluid under the hydrostatic approximation.[8] Such models have also been applied to sea-breezes,[9] horizontal natural convection along a thermally perturbed lower surface,[10] thermal convection between two conducting horizontal boundaries,[11] katabatic flow along a differentially cooled slope,[12] and the flow in typhoons.[13] Solutions of the Navier-Stokes equations for stagnation point flows[14] and von Kármán-Bödewadt vortices[14,15] are also of this special model type.

Unbounded flows consisting of the superposition of (i) a basic flow state in which all of the velocity components vary (at most) linearly with the spatial coordinates (uniform velocity gradients), and (ii) a planar wave with time-dependent wavenumber and amplitude were studied by Kelvin,[16] Lagnado *et al.*,[17] Craik and Criminale,[18] Bayly,[19] Cohen *et al.*,[20] and others. As pointed out by Craik and Criminale,[18] the basic state, planar wave, and their sum are exact solutions of the Navier-Stokes equations. Since the viscous terms in the basic state are identically zero, the basic state is also an exact solution of the Euler equations. This class of solutions was extended by Craik[21] to include Coriolis and buoyancy forces. Admissible time dependencies for the basic states were explored by Craik,[22] though without Coriolis or buoyancy forces. The essentially inviscid instabilities of planar waves in some of these basic states have been investigated for flows without buoyancy or Coriolis forces,[23,24] flows with buoyancy,[25,26] and flows with buoyancy and Coriolis forces.[27]

The present study is concerned with a simple example of a time-dependent basic state of Craik-Criminale type that exhibits a secular growth. The solution is a relative of a solution outlined on page 129 of Craik[22] in which the linear (normal) strain rates are formally periodic but became singular within one period. In the present case, provision for Coriolis and buoyancy forces eliminates the singularity from the linear strain rates for a range of parameter values. The linear strain rates are then periodic and well behaved for all times. However, two of the shear strain rates are forced by the linear strain rates, and their amplitudes grow linearly with time.

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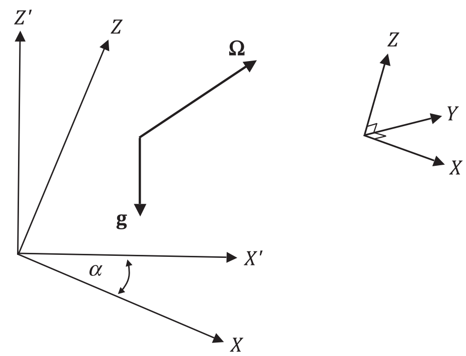

FIG. 1. Cartesian coordinate system (X, Y, Z) used in the analysis. The X and Z axes are obtained by rotating the true horizontal (X') and vertical (Z') axes by an angle α. The Y axis is directed into the page.

II. GOVERNING EQUATIONS

We introduce a Cartesian X, Y, Z coordinate system (Fig. 1) with associated unit vectors $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, whose X–Z plane contains the gravity vector $\mathbf{g}$, and whose X axis is inclined at an angle α to the horizontal direction X' (which is normal to $\mathbf{g}$). Attention is restricted to flows that are two-dimensional in the sense that the dependent variables are independent of Y; however, the three components of the velocity vector $\mathbf{V} = U\,\mathbf{I} + V\,\mathbf{J} + W\,\mathbf{K}$ are generally non-zero. We consider the Boussinesq equations of motion, thermal energy, and mass conservation for flow of a stably stratified fluid in a frame of reference that rotates with angular velocity $\boldsymbol{\Omega} = \Omega_X\mathbf{I} + \Omega_Y\mathbf{J} + \Omega_Z\mathbf{K}$,

$$\frac{\partial B}{\partial T} + (\mathbf{V}\cdot\nabla)B = -N^2\mathbf{K}'\cdot\mathbf{V} + \kappa\,\nabla^2 B, \tag{1}$$

$$\frac{\partial \mathbf{V}}{\partial T} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla\Pi + B\,\mathbf{K}' - 2\boldsymbol{\Omega}\times\mathbf{V} + \nu\,\nabla^2\mathbf{V}, \tag{2}$$

$$\nabla\cdot\mathbf{V} = 0. \tag{3}$$

Here $\nabla \equiv \mathbf{I}\,\partial/\partial X + \mathbf{K}\,\partial/\partial Z$, T is time, and $\mathbf{K}'$ is the unit vector aligned with the Z' axis, which points in the direction opposite to $\mathbf{g}$. The normalized perturbation pressure is $\Pi \equiv (P - P_\infty)/\rho_r$, where P is pressure, P_∞ is (hydrostatic) ambient pressure, and ρ_r is a constant reference density. The buoyancy is $B \equiv g(\theta - \theta_\infty)/\theta_r$, where θ is temperature (if medium is a liquid) or potential temperature (if medium is a gas), θ_r is a constant reference value of θ, $\theta_\infty(Z')$ is an ambient profile of θ, and $g \equiv |\mathbf{g}|$. The Brunt-Väisälä frequency $N \equiv \sqrt{(g/\theta_r)d\theta_\infty/dZ'}$, kinematic viscosity ν and coefficient of thermal diffusivity κ of the fluid are assumed to be constant.

We consider α in the range $0^\circ < \alpha < 90^\circ$, and introduce the non-dimensional variables

$$(x,\ y) \equiv (X,\ Y)\frac{\sin^{3/2}\alpha}{\cos\alpha}\sqrt{\frac{N}{\nu}}, \quad z \equiv Z\sqrt{\frac{N\sin\alpha}{\nu}}, \quad t \equiv T\,N\sin\alpha, \quad b \equiv \frac{B}{N\cos\alpha}\sqrt{\frac{\sin\alpha}{N\nu}},$$
$$(u,\ v) \equiv (U,\ V)\frac{1}{\cos\alpha}\sqrt{\frac{\sin\alpha}{N\nu}}, \quad w \equiv \frac{W}{\sqrt{\nu\,N\sin\alpha}}, \quad \pi \equiv \frac{\Pi}{N\nu}\frac{\sin\alpha}{\cos^2\alpha}, \tag{4}$$

in terms of which (1)–(3) expand out as

$$\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial x} + w\frac{\partial b}{\partial z} = u - w + \frac{1}{\mathrm{Pr}}\left(\tan^2\alpha\,\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial z^2}\right), \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{\partial \pi}{\partial x} + \sigma\,v - \gamma\,w - b + \tan^2\alpha\,\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}, \tag{6}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} = -\frac{\partial \pi}{\partial y} - \sigma\, u + \varepsilon\, w + \tan^2\alpha\, \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}, \tag{7}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = \cot^2\alpha \left(-\frac{\partial \pi}{\partial z} + b + \gamma\, u - \varepsilon\, v \right) + \tan^2\alpha\, \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}, \tag{8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{9}$$

where Pr $\equiv \nu/\kappa$ is the Prandtl number, and

$$\varepsilon \equiv \frac{2\Omega_X}{N\cos\alpha}, \qquad \gamma \equiv \frac{2\Omega_Y}{N\cos\alpha}, \qquad \sigma \equiv \frac{2\Omega_Z}{N\sin\alpha}, \tag{10}$$

are dimensionless parameters.

Attention is restricted to the special class of flows with velocity components given by

$$u = u_x x + u_z z, \quad v = v_x x + v_z z, \quad w = -u_x z, \tag{11}$$

where the linear strain rate u_x and the shear strain rates u_z, v_x, v_z vary in time but not space. The form of w guarantees that (9) is satisfied. Although it is permissible to include spatially constant variables u_0, v_0, w_0 in (11), we have chosen not to do so because such variables would not affect the behavior of u_x, u_z, v_x, v_z, which is the focus of our analysis.

The forms of b and π consistent with (11) and the governing Eqs. (5)–(9) are

$$b = b_0 + b_x x + b_z z, \quad \pi = \frac{1}{2}\pi_{xx} x^2 + \pi_{xz} x z + \frac{1}{2}\pi_{zz} z^2, \tag{12}$$

where b_x, b_z, π_{xz}, and π_{zz} are functions of time but not of space, and π_{xx} is constant in space and time (time dependence for π_{xx} is also permissible but will not be considered here). It can be shown that since u_0, v_0, and w_0 are taken to be zero, pressure terms that vary linearly with the spatial coordinates drop from consideration.

Applying (11) and (12) in (5)–(9), and collecting terms in common powers of x and z, we obtain

$$\frac{db_z}{dt} + u_z b_x - u_x b_z = u_z + u_x, \tag{13}$$

$$\frac{db_x}{dt} + u_x b_x = u_x, \tag{14}$$

$$\frac{du_z}{dt} = -\pi_{xz} + \sigma\, v_z + \gamma\, u_x - b_z, \tag{15}$$

$$\frac{du_x}{dt} + u_x^2 = -\pi_{xx} + \sigma\, v_x - b_x, \tag{16}$$

$$\frac{dv_z}{dt} + u_z v_x - u_x v_z = -\sigma\, u_z - \varepsilon\, u_x, \tag{17}$$

$$\frac{dv_x}{dt} + u_x v_x = -\sigma\, u_x, \tag{18}$$

$$-\frac{du_x}{dt} + u_x^2 = \cot^2\alpha\, (-\pi_{zz} + b_z + \gamma\, u_z - \varepsilon\, v_z), \tag{19}$$

$$\pi_{xz} = b_x + \gamma\, u_x - \varepsilon\, v_x. \tag{20}$$

Since the velocity and buoyancy fields vary linearly with x and z, the diffusion/viscous terms are identically zero. No approximations were made in proceeding from (1)–(3) to (13)–(20).

It should be noted that if $\alpha = 0^\circ$ or $\alpha = 90^\circ$, we can non-dimensionalize variables using a version of (4) in which α does not appear. Application of (11) and (12) then leads to equations similar to (13)–(20), though containing fewer terms. The solutions (not shown) are similar to those obtained in Sec. III for $0^\circ < \alpha < 90^\circ$, the largest difference being that with $\alpha = 0^\circ$ the b_x variable is forced by, but does not affect, u_x or v_x.

We identify two types of steady-state solutions of (13)–(20) corresponding to flows in which fluid parcels are (i) not accelerating, $(\mathbf{V}\cdot\nabla)\mathbf{V} = 0$, or (ii) accelerating, $(\mathbf{V}\cdot\nabla)\mathbf{V} \neq 0$. The former type of solutions describe flows with $u = 0$, $w = 0$ in the following balance:

$$\begin{aligned} &u_x = 0, \quad u_z = 0, \quad \pi_{xx} = \sigma\, v_x - b_x, \quad \pi_{zz} = b_z - \varepsilon\, v_z, \\ &\pi_{xz} = \sigma\, v_z - b_z, \quad \pi_{xz} = b_x - \varepsilon\, v_x. \end{aligned} \tag{21}$$

Eliminating π_{xz} from the last two equations in (21) yields the y-component vorticity equation,

$$0 = \varepsilon\, v_x + \sigma\, v_z - b_x - b_z. \tag{22}$$

Equation (22) expresses a balance between the baroclinic generation of y-component vorticity [$-b_x$ and $-b_z$ combine to form the horizontal (X') derivative of b], tilting by v of the x-component reference frame vorticity into the y-direction ($\varepsilon\, v_x$), and tilting by v of the z-component reference frame vorticity into the y direction ($\sigma\, v_z$). One may freely prescribe values for any three of v_x, v_z, b_x, b_z, and recover the remaining variable from (22). The pressure variables then follow from (21).

The accelerating steady-state solutions describe a flow in which fluid parcels are linearly strained ($u_x \neq 0$) and accelerated by the pressure gradient force,

$$\begin{aligned} &b_x = 1, \qquad b_z = -1, \qquad v_x = -\sigma, \qquad v_z = \varepsilon, \qquad \pi_{xx} = -u_x^2 - 1 - \sigma^2, \\ &\pi_{zz} = -u_x^2 \tan^2\alpha + \gamma u_z - 1 - \varepsilon^2, \qquad \pi_{xz} = \gamma u_x + 1 + \sigma\,\varepsilon. \end{aligned} \tag{23}$$

Written in dimensional form and cast in terms of θ, the equations for b_x and b_z show that θ is constant (continual straining pulls apart θ surfaces, leading to a vanishing gradient of θ). The equations for v_x and v_z show that the z- and x-components of the absolute vorticity (which are proportional to $v_x + \sigma$ and $-v_z + \varepsilon$, respectively) are zero. One may freely prescribe values for u_x and u_z, and then recover the pressure variables from (23).

The remainder of our study is concerned with unsteady solutions of (13)–(20). Equations (14), (16), and (18) form a closed nonlinear system for u_x, v_x, and b_x and can be solved first. Those variables include the two non-zero linear strain rates, $\partial u/\partial x = -\partial w/\partial z = u_x$, (the third linear strain rate is zero) and (twice) one of the shear strain rates ($\partial u/\partial y + \partial v/\partial x = v_x$). The projection of the streamlines associated with this system on the xz plane are hyperbolas. As we will see, u_x is generally oscillatory, and so the flow associated with it can be described as an alternating convergent-divergent flow (by divergence we mean a positive value of the two-dimensional divergence of the velocity field, $\partial u/\partial x + \partial v/\partial y$, which is just u_x in our case). For brevity, we will refer to u_x, v_x, and b_x as divergent mode variables. Once the divergent mode variables have been obtained, we can recover π_{xz} as a residual from (20). Then, with π_{xz} and the divergent mode variables known, the linear system (13), (15), and (17) can be solved for u_z, v_z, and b_z. We will refer to these latter variables as the non-divergent mode variables. Lastly, π_{zz} can be recovered from (19).

III. ANALYTIC SOLUTION

A. Divergent mode

To solve (14), (16), and (18), it is convenient to introduce a new variable λ defined through the Riccati substitution (e.g., Craik,[22] Shapiro,[4] and Leblanc[26])

$$u_x = \frac{1}{\lambda}\frac{d\lambda}{dt}. \tag{24}$$

Since replacing λ by any non-zero constant times λ leaves (24) unchanged, we are free to normalize λ in any convenient manner. Applying (24) in (14), multiplying the resulting equation through by an integrating factor (which is λ) and then integrating, we obtain

$$b_x = 1 + A/\lambda, \tag{25}$$

where A is a constant. Similarly, applying (24) in (18), multiplying through by λ and then integrating yields

$$v_x = -\sigma + B/\lambda, \tag{26}$$

where B is a constant. Application of (24)–(26) in (16) produces

$$\frac{d^2\lambda}{dt^2} + (1 + \sigma^2 + \pi_{xx})\lambda = \sigma B - A. \tag{27}$$

If $\pi_{xx} < -1 - \sigma^2$ (pressure gradient force $-x\pi_{xx}$ points away from pressure maximum at $x = 0$), the general solution of (27) is the sum of a constant and a linear combination of growing and decaying exponential terms. As $t \to \infty$, the growing term dominates (except for special cases where that term is identically zero), $|\lambda| \to \infty$, and (24)–(26) indicate that $b_x \to 1$, $v_x \to -\sigma$, while u_x approaches a non-zero constant. Thus, as $t \to \infty$, the flow approaches the accelerating steady state described by (23). However, if the initial conditions are such that $\lambda(t)$ passes through zero at a finite time, the solution becomes singular.

In the remainder of the study we restrict attention to values $\pi_{xx} > -1 - \sigma^2$. Here π_{xx} contributes to a pressure gradient force that points toward a pressure minimum at $x = 0$ if $\pi_{xx} > 0$, and points away from a pressure maximum at $x = 0$ if $-1 - \sigma^2 < \pi_{xx} < 0$. The general solution of (27) is

$$\lambda(t) = q + \cos(\omega t), \tag{28}$$

where

$$\omega \equiv \sqrt{1 + \sigma^2 + \pi_{xx}}, \qquad q \equiv (\sigma B - A)/\omega^2, \tag{29}$$

and an arbitrary phase angle has been set to zero. It can be noted that we have normalized λ in (28) so that the amplitude of its fluctuating part is unity. The dimensional frequency ω_{dim} corresponding to ω in (29) is of the form $\omega_{\text{dim}} = [N^2 \sin^2\alpha + 4\Omega_Z^2 + \partial^2\Pi/\partial X^2]^{1/2}$, which is the frequency of inertial-gravity waves modified by a factor of $\partial^2\Pi/\partial X^2$.

Applying (28) in (24)–(26) yields

$$u_x = -\frac{\omega \sin(\omega t)}{q + \cos(\omega t)}, \tag{30}$$

$$b_x = 1 + \frac{A}{q + \cos(\omega t)}, \tag{31}$$

$$v_x = -\sigma + \frac{B}{q + \cos(\omega t)}. \tag{32}$$

With the phase angle set to zero, $t = 0$ corresponds to the phase when $u_x = 0$.

Consider arbitrary initial values of v_x and b_x. If $b_x(0) - \sigma v_x(0) + \pi_{xx} \neq 0$, A and B can be expressed as

$$A = \frac{\omega^2[b_x(0) - 1]}{b_x(0) - \sigma v_x(0) + \pi_{xx}}, \qquad B = \frac{\omega^2[v_x(0) + \sigma]}{b_x(0) - \sigma v_x(0) + \pi_{xx}}, \tag{33}$$

and q becomes

$$q = \frac{\omega^2}{b_x(0) - \sigma v_x(0) + \pi_{xx}} - 1. \tag{34}$$

If $b_x(0) - \sigma v_x(0) + \pi_{xx} = 0$, then (16) [with $u_x(0) = 0$] yields $(du_x/dt)_{t=0} = 0$, while (14) and (18) yield $(db_x/dt)_{t=0} = (dv_x/dt)_{t=0} = 0$. Further differentiating (14), (16), and (18) and evaluating

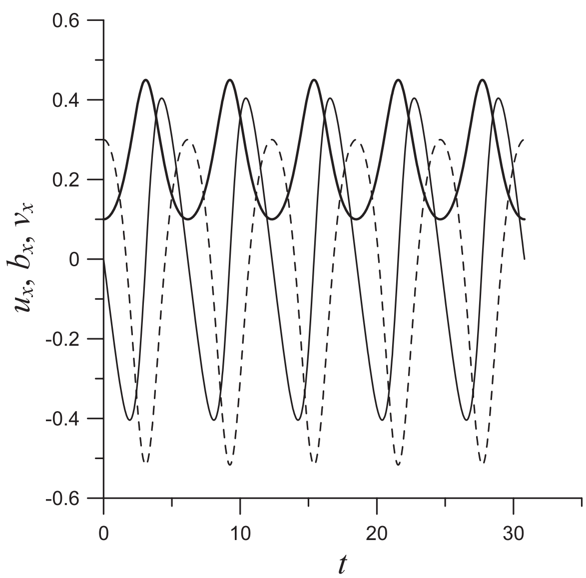

FIG. 2. Divergent mode variables u_x (solid line), v_x (bold line), and b_x (dashed line) versus time t for the case with $\sigma = 0.2$, $\varepsilon = 0.4$, $\pi_{xx} = 0$, $u_x(0) = 0$, $v_x(0) = 0.1$, and $b_x(0) = 0.3$ ($\omega \approx 1.019$, $q \approx 2.714$). All quantities are non-dimensional.

the results at $t = 0$, shows that all higher order time derivatives of u_x, v_x, and b_x are also zero. We thus obtain the steady state (21).

To ensure that the denominators in (30)–(32) do not vanish, we restrict q to values such that $|q| > 1$. In view of (34), the condition for (30)–(32) to be free of singularities is

$$b_x(0) - \sigma v_x(0) + \pi_{xx} < \omega^2/2, \qquad (|q| > 1)\,. \tag{35}$$

An example of a divergent mode oscillation is presented in Fig. 2.

B. Non-divergent mode

Applying (24)–(26) and (20) in (13), (15), and (17), and introducing the new variables

$$b^* \equiv b_z + 1, \qquad v^* \equiv v_z - \varepsilon, \qquad u^* \equiv u_z, \tag{36}$$

the non-divergent mode equations reduce to

$$\frac{db^*}{dt} - \frac{b^*}{\lambda}\frac{d\lambda}{dt} + u^*\frac{A}{\lambda} = 0, \tag{37}$$

$$\frac{du^*}{dt} + \frac{A - \varepsilon B}{\lambda} - \sigma v^* + b^* = 0, \tag{38}$$

$$\frac{dv^*}{dt} - \frac{v^*}{\lambda}\frac{d\lambda}{dt} + u^*\frac{B}{\lambda} = 0. \tag{39}$$

These equations are linear but contain time-dependent (periodic) coefficients. Because an inhomogeneous term $(A - \varepsilon B)/\lambda$ appears in (38), a non-divergent motion is inevitable. We also note that the terms involving the Y-component of the reference frame vorticity γ in (15) and (20) have canceled out. No such terms appear in (37), (38) or (39). Thus, the only effect of γ is the modification of the pressure variables π_{zz} and π_{xz} through (19) and (20).

We get a relation between v^* and b^* by subtracting A times (39) from B times (37), and dividing the resulting equation by λ. Integrating and rearranging the result provides

$$Bb^* - Av^* = C\lambda, \tag{40}$$

where C is a constant of integration. To get a single equation for b^*, differentiate (37), and use (37), (38), and (40) to eliminate u^* and v^* from the resulting equation to obtain

$$\frac{d^2b^*}{dt^2} + b^*\left(\frac{\sigma\, B - A}{\lambda} - \frac{1}{\lambda}\frac{d^2\lambda}{dt^2}\right) = \sigma\, C + \frac{A(A-\varepsilon B)}{\lambda^2}. \tag{41}$$

In view of (27) and (29), the coefficient of the b^* term in (41) is just ω^2 and we arrive at

$$\frac{d^2b^*}{dt^2} + \omega^2 b^* = \sigma C + \frac{A(A-\varepsilon B)}{\lambda^2}. \tag{42}$$

The homogeneous part of (42) describes an undamped linear harmonic oscillator with natural frequency ω. In view of (28), the inhomogeneous term in (42) provides a forcing with the same frequency. We thus anticipate that the solution may exhibit secular growth (see pages 544 and 545 of Bender and Orszag[28] for a classical example of resonance and secular behavior in a forced harmonic oscillator). The method of variation of parameters yields the general solution of (42) as

$$b^* = \frac{\sigma C}{\omega^2} + D\cos(\omega t) + E\sin(\omega t) + \frac{A(A-\varepsilon B)}{\omega}\left[\sin(\omega t)I(t) - \cos(\omega t)J(t)\right], \tag{43}$$

where

$$J(t) \equiv \int_0^t \frac{\sin(\omega\tilde{t})}{\lambda^2(\tilde{t})}d\tilde{t}, \qquad I(t) \equiv \int_0^t \frac{\cos(\omega\tilde{t})}{\lambda^2(\tilde{t})}d\tilde{t}, \tag{44}$$

and D and E are constants. The integrals $J(t)$ and $I(t)$ are evaluated as

$$J(t) = \frac{1}{\omega}\left[\frac{1}{q+\cos(\omega t)} - \frac{1}{q+1}\right], \tag{45}$$

$$I(t) = \frac{q}{\omega(q^2-1)}\frac{\sin(\omega t)}{\lambda} - \frac{2}{\omega(q^2-1)^{3/2}}\tan^{-1}\left[\frac{q-1}{\sqrt{q^2-1}}\tan\left(\frac{\omega t}{2}\right)\right], \tag{46}$$

where q is defined through (29). To obtain $I(t)$ as a continuous function of time, we must account for the multi-valued nature of the $\tan^{-1}$ function. Careful consideration of the argument of $\tan^{-1}$ in (46) shows that its principal value jumps by a factor of π periodically, at times $t = (2k-1)\pi/\omega$ ($k = 1, 2, 3\ldots$). We must therefore increment the principal value by $k\pi$ at those times to counter the jumps. The sense of a jump (positive or negative) depends only on the sign of $q-1$, and is thus the same for any particular flow case. Accordingly, the envelope of $I(t)$ increases linearly with time and provides the anticipated secular behavior.

Applying (43) and (29) in (40), we obtain v^* as

$$v^* = \frac{C}{\omega^2} + \left[\frac{BD-C}{A} - \frac{BA-\varepsilon B^2}{\omega}J(t)\right]\cos(\omega t) + \left[\frac{BE}{A} + \frac{BA-\varepsilon B^2}{\omega}I(t)\right]\sin(\omega t). \tag{47}$$

Applying (43) and (44) in (37), we obtain u^* as

$$u^* = \left[\frac{q\omega D}{A} - \frac{\sigma C}{\omega A} - q(A-\varepsilon B)J(t)\right]\sin(\omega t) - \left[\frac{\omega E}{A} + (A-\varepsilon B)I(t)\right]\left[1 + q\cos(\omega t)\right]. \tag{48}$$

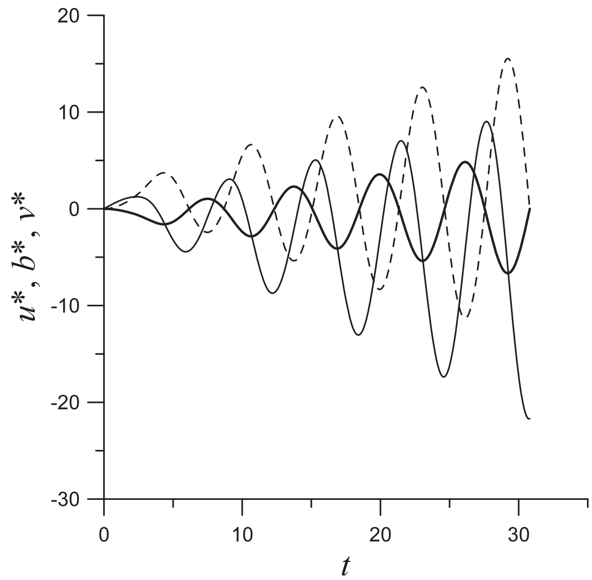

FIG. 3. Non-divergent mode variables u^* (solid line), v^* (bold line), and b^* (dashed line) versus time t corresponding to the divergent mode displayed in Fig. 2, and the initial values $u^*(0) = 0$, $v^*(0) = 0$, and $b^*(0) = 0$. All quantities are non-dimensional.

The constants C, D, and E are related to the initial values of u^*, v^*, and b^* by

$$E = -\frac{Au^*(0)}{\omega(1+q)}, \quad C = \frac{Bb^*(0) - Av^*(0)}{q+1}, \quad D = \frac{\sigma A v^*(0) - \sigma B b^*(0)}{\omega^2(q+1)} + b^*(0). \tag{49}$$

The secular parts of the solutions, involving $I(t)$ in (43), (47), and (48), are independent of E, C, and D, and are thus independent of the initial values of the non-divergent mode variables.

The secular behavior of the non-divergent mode variables is evident in the example depicted in Fig. 3. The depicted mode was forced by the divergent mode shown in Fig. 2.

It can be demonstrated that if the divergent mode is not present or is in a steady state given by (21) then the equations for the non-divergent mode admit free oscillation solutions with frequency defined in (29). In other words, the natural frequencies of the non-divergent and divergent modes are the same. Thus, if an oscillatory divergent mode is present, its frequency matches the natural frequency of the non-divergent mode and the non-divergent solution exhibits secular growth.

However, (43), (47), and (48) show that the secular term drops from the solutions in the special case where $A - \varepsilon B = 0$. To better understand this special case, we eliminate π_{xz} from (15) and (20), obtaining the y-component vorticity equation

$$\frac{du_z}{dt} = \varepsilon\, v_x + \sigma\, v_z - b_x - b_z. \tag{50}$$

If an oscillatory divergent mode is present, $\varepsilon\, v_x$ and $-b_x$ oscillate with frequency ω and generally promote secular growth. However, secular growth would not be expected in the special case where the oscillatory part of $\varepsilon\, v_x$ [which (32) yields as $\varepsilon\,(v_x + \sigma)$] cancels with the oscillatory part of $-b_x$ [which (31) yields as $-(b_x - 1)$], that is, if

$$\varepsilon(\, v_x + \sigma) - (b_x - 1) = 0. \tag{51}$$

In view of (31) and (32), the ratio$(v_x + \sigma)/(b_x - 1)$ is independent of time. Thus, if (51) is true at one time, it is true at all times. Applying (33) for A and B in the condition for no secular growth, $A - \varepsilon B = 0$, yields (51) evaluated at $t = 0$. Thus, $A - \varepsilon B = 0$ corresponds to the case where (51) is satisfied at all times.

IV. STREAMLINES AND TRAJECTORIES

The stream function ψ, defined through $u = \partial\psi/\partial z$ and $w = -\partial\psi/\partial x$, is obtained from (11) as

$$\psi = u_x(t)x\,z + \frac{1}{2}u_z(t)z^2. \tag{52}$$

Setting ψ to constant values in (52) yields the projections of streamlines on the xz plane. For small z, these curves are hyperbolas and the flow is dominated by an alternating in-up and down-out oscillation. For large z, the curves are largely independent of x and the flow is dominated by a shear flow in which u oscillates (and grows) in time and varies linearly with z.

To obtain the projections of streamlines on the yz plane, we integrate the equation for the slope of those curves, $dz/dy = w/v = -u_x z/(v_x x + v_z z)$, obtaining

$$y = -x\frac{v_x(t)}{u_x(t)}\ln|z| - \frac{v_z(t)}{u_x(t)}z + a, \tag{53}$$

where a is a constant of integration.

The trajectories satisfy

$$\frac{dx}{dt} = u_x(t)x(t) + u_z(t)z(t), \tag{54}$$

$$\frac{dy}{dt} = v_x(t)x(t) + v_z(t)z(t), \tag{55}$$

$$\frac{dz}{dt} = -u_x(t)z(t). \tag{56}$$

Applying (24) in (56) and integrating the result yields the z-component of the trajectory as

$$z(t) = z(0)\frac{\lambda(0)}{\lambda(t)}, \tag{57}$$

where $\lambda(t)$ is known from (28). Applying (57) and (24) in (54), dividing the resulting equation by λ, then integrating and rearranging the result provides

$$x(t) = \frac{x(0)}{\lambda(0)}\lambda(t) + z(0)\lambda(0)\lambda(t)\int_0^t \frac{u_z(\tilde{t})}{\lambda^2(\tilde{t})}d\tilde{t}. \tag{58}$$

To evaluate the integral, divide (37) by λ (recall $u^* = u_z$) and integrate the result to arrive at

$$\int_0^t \frac{u_z(\tilde{t})}{\lambda^2(\tilde{t})}d\tilde{t} = \frac{1}{A}\left[\frac{b^*(0)}{\lambda(0)} - \frac{b^*(t)}{\lambda(t)}\right]. \tag{59}$$

Applying (59) in (58) yields the x-component of the trajectory:

$$x(t) = \left[\frac{x(0)}{\lambda(0)} + \frac{z(0)}{A}b^*(0)\right]\lambda(t) - z(0)\frac{\lambda(0)}{A}b^*(t), \tag{60}$$

where $b^*(t)$ is known from (43).

From (57), we see that $z(t)$ depends only on the divergent mode, and is thus periodic. In contrast, (60) shows that both periodic and amplifying functions contribute to $x(t)$. At times t for which sin $(\omega t) = 0$, the amplifying part of the solution corresponding to $\sin(\omega t)I(t)$ in (43) vanishes, leaving only periodic contributions. This happens twice per cycle. The projection of the trajectory curve on the xz plane thus consists of ever-widening (in x) loops that pass through the same two locations each cycle. This behavior is illustrated by the sample trajectory displayed in Fig. 4(a).

To obtain $y(t)$, apply the solutions for $x(t)$, $z(t)$, $v_x(t)$, and $v_z(t)[=v^*(t)+\varepsilon]$ in (55), and use (40) to simplify the resulting expression. With the right-hand side of (55) now expressed as an explicit function of time, it is straightforward (though tedious) to integrate the equation analytically.

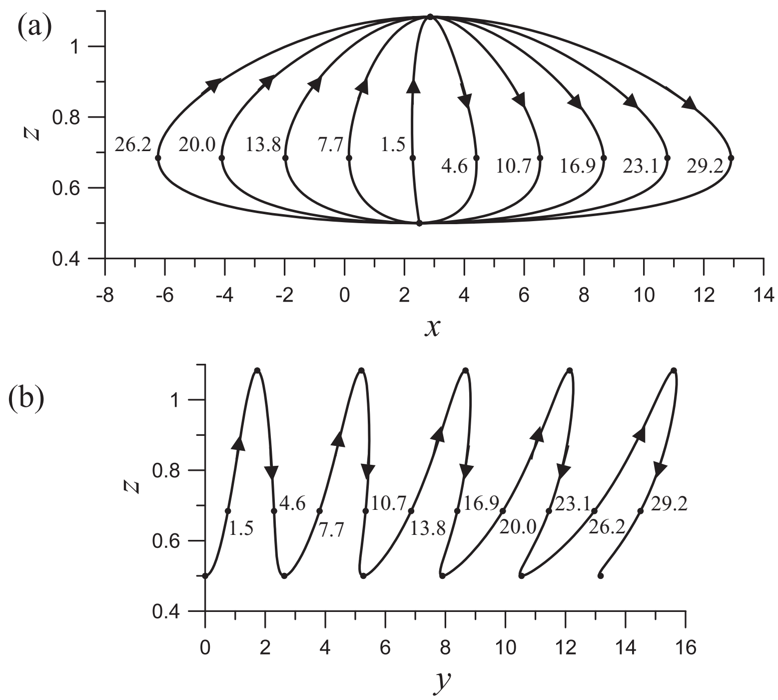

FIG. 4. A sample trajectory for the flow case considered in Figs. 2 and 3. The projections of the trajectory curve on the xz and yz planes are shown in panels (a) and (b), respectively. The fluid parcel is initially at $x(0) = 2.5$, $y(0) = 0$, and $z(0) = 0.5$. Parcel locations at selected times are as indicated. Scales of x and y are the same. All quantities are non-dimensional.

The terms involving I and J offer the most difficulty, but by making use of (44)–(46), they can be integrated by parts. We thus arrive at

$$\begin{aligned} y(t) = y(0) &+ \left[(B - \sigma q)\left(\frac{x(0)}{\lambda(0)} + \frac{z(0)}{A} b^*(0)\right) + C\lambda(0)\frac{z(0)}{A}\left(\frac{\sigma^2}{\omega^2} - 1\right)\right] t \\ &+ \frac{\sigma}{\omega}\left[D\lambda(0)\frac{z(0)}{A} - \frac{x(0)}{\lambda(0)} - \frac{z(0)}{A} b^*(0) - \frac{z(0)}{\omega^2}\lambda(0)\frac{A - \varepsilon B}{q^2 - 1}\right] \sin(\omega t) \\ &+ \frac{2z(0)\lambda(0)}{\omega(q^2 - 1)^{1/2}}\left[\varepsilon + \sigma\frac{(A - \varepsilon B)}{\omega^2(q^2 - 1)}\lambda(t)\right] \tan^{-1}\left[\frac{q - 1}{\sqrt{q^2 - 1}} \tan\left(\frac{\omega t}{2}\right)\right] \\ &+ E\frac{\sigma z(0)}{\omega A}\lambda(0)\left[1 - \cos(\omega t)\right]. \end{aligned} \tag{61}$$

An example of the projection of a trajectory curve on the yz plane is shown in Fig. 4(b). The tendency of the curve to lean to the right at later times is a consequence of the intensifying v_z shear strain rate.

V. SUMMARY

Simple exact solutions of the Boussinesq equations of motion, thermal energy, and mass conservation are obtained for an oscillatory flow motion in an unbounded stably stratified rotating fluid. The flow is considered two-dimensional in the sense that all flow variables are independent of the Cartesian Y coordinate; however, the three velocity components are generally non-zero. The flow is characterized by velocity and buoyancy gradients that vary with time but are spatially uniform—the basic state of Craik-Criminale flows. These constraints reduce the governing partial differential equations to a set of ordinary differential equations. The solution is obtained analytically by first solving a sub-set of these equations for a nonlinear divergent mode, and then solving a set of linear inhomogeneous ordinary differential equations for a non-divergent mode. The solution describes a periodic free oscillation of the divergent flow mode and the secular growth that this oscillation provokes in the non-divergent flow mode.

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