



On the limiting effect of the Earth's rotation on the depth of a stably stratified boundary layer

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Two alternative depth scales have been proposed for the case of a stably stratified boundary layer (SBL) where static stability is due to the surface buoyancy flux B_s . Kitaigorodskii in 1960 assumed that the Earth's rotation is no longer important as static stability becomes strong and that the SBL depth scales with the Obukhov length $L = -u_*^3/B_s$, u_* being the surface-friction velocity. Zilitinkevich in 1972 proposed an alternative scale, $(u_*L/|f|)^{1/2}$, which depends on the Coriolis parameter f no matter how strong the static stability. Similarly, two alternative depth scales have been proposed for the case of a SBL dominated by static stability at its outer edge with buoyancy frequency N . The depth scale u_*/N introduced by Kitaigorodskii and Joffe in 1988 does not depend on the Coriolis parameter, whereas the Pollard, Rhines and Thompson scale $u_*/|Nf|^{1/2}$ introduced in 1973 does.

In the present article, the above formulations for the SBL depth are shown to be consistent with the budgets of momentum and of turbulence kinetic energy in the SBL. Furthermore, it is demonstrated that in the case of sufficiently strong static stability the alternative depth-scale formulations represent particular cases of more general power-law expressions. For a SBL dominated by the surface buoyancy flux, the generalized depth scale is given by $L(|f|L/u_*)^{-\gamma}$. For a SBL dominated by outer-edge static stability, the generalized scale is $(u_*/N)(|f|/N)^{-\delta}$. The exponents γ and δ lie in the range from 0 to 1/2. With $\gamma = 1/2$ and $\delta = 1/2$, these expressions yield the Zilitinkevich scale and the Pollard *et al.* scale, respectively. In the limits $\gamma = 0$ and $\delta = 0$, the SBL depth scales cease to depend on the Coriolis parameter in their explicit form and the formulations proposed by Kitaigorodskii and by Kitaigorodskii and Joffe, respectively, are recovered.

Simple dimensionality arguments are not sufficient to determine γ and δ . To do this would require an exact solution to equations governing the structure of mean fields and turbulence in the SBL. Since such a solution is not known, the exponents should be evaluated from experimental data. Available data from observations and from large-eddy simulations are uncertain. They do not make it possible to evaluate γ and δ to adequate accuracy and to decide conclusively between the alternative formulations for the SBL depth. As regards practical applications, previously proposed multi-limit formulations based on the above depth scales with γ and δ in the range from 0 to 1/2 are expected to give similar results for stability conditions typical of the atmospheric and oceanic SBLs, provided the disposable dimensionless coefficients in the multi-limit formulations are appropriately tuned.

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1. Introduction

A stably stratified boundary layer (SBL) is a matter of keen interest in geophysical fluid dynamics. Although a lot of studies have been devoted to the SBL, there are still many controversial issues that need to be discussed further and eventually resolved. One such issue is the equilibrium SBL depth, that is the depth in a quasi-steady state to which the SBL evolves in response to external forcing. In the present article, this issue is reconsidered with an emphasis on the effect of the Earth's rotation on the equilibrium depth of a stably stratified barotropic boundary layer.

It should be stressed that the discussion in the present article is limited to the depth-scale formulations for a quasi-steady-state barotropic SBL. Other aspects of the real-world SBL, such as the effect of the horizontal components of the angular velocity of the Earth's rotation and the Ekman-layer rectification phenomenon (Zikanov *et al.*, 2003; McWilliams and Huckle, 2006; McWilliams *et al.*, 2009), although very important, are beyond the scope of the present study. However, even highly idealized SBL archetypes, such as a barotropic quasi-steady-state SBL, are of great significance. Apart from their academic utility, they are widely used in applications. For example, the equilibrium SBL depth is one of the key parameters in pollution dispersion studies.

A number of formulations for the equilibrium SBL depth h have been proposed to date (see discussions in Zilitinkevich and Mironov, 1996, hereafter ZM96; Zilitinkevich *et al.*, 2002, hereafter ZBRSLC02; Zilitinkevich and Esau, 2003; Hess, 2004; Zilitinkevich *et al.*, 2007). The major formulations are summarized in Table I, where u_* is the surface friction velocity, B_s is the surface buoyancy flux, f is the Coriolis parameter, N is the buoyancy frequency at the SBL outer edge and $L = -u_*^3/B_s$ is the Obukhov (1946) length scale (we omit the von Kármán constant κ from the expression for L).

As seen from Table I, there are two alternative depth scales for an SBL dominated by a stabilizing surface buoyancy flux. Kitaigorodskii (1960) held the viewpoint that in the case of strong static stability the Earth's rotation is no longer important and the SBL depth scales with the Obukhov length. Zilitinkevich (1972) proposed an alternative formulation, where the SBL depth depends on the Coriolis parameter no matter how strong the static stability. Similarly, the Kitaigorodskii and Joffre (1988) depth scale for an SBL affected by static stability at its outer edge (we will also refer to such an SBL as the imposed-stability-dominated SBL) does not depend on the Coriolis parameter, whereas the Pollard *et al.* (1973) depth scale does.

The results from earlier studies were summarized by ZM96, who concluded that the Rossby and Montgomery (1935), Kitaigorodskii (1960) and Kitaigorodskii and Joffre (1988) scales hold in the limiting cases of a truly neutral rotating boundary layer, a surface-flux-dominated SBL and an imposed-stability-dominated SBL, respectively. The Zilitinkevich (1972) and Pollard *et al.* (1973) scales were found to describe the intermediate regimes, where the effects of rotation and stratification are roughly equally important. A multi-limit SBL-depth formulation was proposed in ZM96 that accounts for the combined effects of rotation, surface buoyancy flux and static stability at the SBL outer edge. It

reads

$$\left(\frac{h|f|}{C_n u_*}\right)^2 + \frac{h}{C_s L} + \frac{hN}{C_i u_*} + \frac{h|f|^{1/2}}{C_{sr}(u_* L)^{1/2}} + \frac{h|Nf|^{1/2}}{C_{ir} u_*} = 1, \quad (1)$$

where C_n , C_s , C_i , C_{sr} and C_{ir} are dimensionless constants (we use the original ZM96 notation). Equation (1) shows that h ceases to depend on f in the case of strong static stability, i.e. when L or/and u_*/N is small compared with $u_*/|f|$. A simplified version of Eq. (1) that does not incorporate the intermediate scales, represented by the last two terms on the left-hand side (l.h.s.), was favourably tested against data from large-eddy simulations (LES) and from measurements in the atmospheric and benthic SBLs.

The problem of the equilibrium SBL depth was reconsidered by ZBRSLC02. They concluded that the appropriate depth scales for boundary layers dominated by the surface buoyancy flux and by static stability at the outer edge of the boundary layer are the Zilitinkevich (1972) scale and the Pollard *et al.* (1973) scale, respectively, and proposed the following multi-limit formulation for the equilibrium SBL depth:

$$\left(\frac{h|f|}{C_R u_*}\right)^2 + \frac{h^2|f|}{C_S^2 u_* L} + \frac{C_{uN} h^2 N |f|}{C_S^2 u_*^2} = 1, \quad (2)$$

where C_R , C_S and C_{uN} are dimensionless constants (the original ZBRSLC02 notation is retained). Equation (2), which was favourably tested against observational and LES data in ZBRSLC02, shows that h depends on f no matter how strong the static stability.

Worthy of mention is the depth scale $u_*^2/|B_s N|^{1/2}$ obtained by McWilliams *et al.* (2009) by considering the oceanic SBL affected by both the surface buoyancy flux and the stable density stratification beneath the boundary layer. However, as the authors of *op. cit.* state, this scale 'seems unlikely to be a physically common or important regime except in extreme conditions of heating and stratification'.

In the subsequent text, we examine the alternative formulations for the SBL depth given in Table I from the standpoint of their consistency with the budgets of turbulence kinetic energy (TKE) and of momentum in the boundary layer (sections 2 and 3, respectively). We demonstrate (section 4) that the alternative formulations can all be derived on the basis of TKE-budget and momentum-budget considerations, although different assumptions should be made on the way. In section 5, we propose generalized power-law formulations that incorporate the alternative SBL depth scales as particular cases. Then (section 6) we test Eq. (1) (more specifically, its reduced form without the last two terms on the l.h.s.) and Eq. (2) against numerical and observational data. Results of the study are summarized in section 7.

2. Turbulence kinetic energy budget considerations

The steady-state TKE balance equation is

$$-(\tau_x \partial u / \partial z + \tau_y \partial v / \partial z) + B - \partial F / \partial z - \epsilon = 0, \quad (3)$$

where z is height, u and v are the velocity components along the x and y horizontal axes, respectively, τ_x and τ_y are the x

Table I. The equilibrium boundary-layer depth scales.

Truly neutral boundary layer	$u_*/ f $ (Rossby and Montgomery, 1935)	
Surface-flux-dominated SBL	$-u_*^3/B_s = L$ (Kitaigorodskii, 1960)	$u_*^2/ B_s f ^{1/2} = (u_* L/ f)^{1/2}$ (Zilitinkevich, 1972)
Imposed-stability-dominated SBL	u_*/N (Kitaigorodskii and Joffre, 1988)	$u_*/ Nf ^{1/2}$ (Pollard, Rhines and Thompson, 1973)

and y components of the kinematic turbulent momentum flux, B is the vertical buoyancy flux, F is the vertical TKE flux due to the third-order velocity correlation and the velocity-pressure correlation and ϵ is the TKE dissipation rate.

Integrating Eq. (3) over the boundary layer, we obtain

$$u_*^2 U_g + \int_0^h B dz - F_h - \int_0^h \epsilon dz = 0, \tag{4}$$

where U_g is the geostrophic wind component along the x -axis and F_h is the energy flux at the boundary-layer outer edge $z = h$. The first term on the l.h.s. of Eq. (4), which represents the integral shear production of TKE, is obtained using the steady-state momentum balance, see Eqs (7) and (8) below, subject to no-slip boundary condition at the surface and the conditions $u = U_g$ and $v = V_g$ at the SBL top (V_g being the geostrophic wind component along the y -axis), taking the x -axis aligned with the surface stress and assuming that the stress at the SBL top is negligible (see Zilitinkevich, 1989, and ZM96). The energy flux F_h is due to internal gravity waves that transfer energy from the boundary layer to the stably stratified fluid aloft. The energy flux at the surface $z = 0$ is neglected. Its inclusion presents no principal difficulties, but is not necessary in the present context as the end result remains unchanged (see discussion in ZM96).

The similarity arguments suggest that the expression for the integral TKE dissipation should incorporate terms whose functional form is the same as that of the shear production and the buoyancy destruction terms in the TKE budget, which are the first and the second terms on the l.h.s. of Eq. (4), respectively. Then $\int_0^h \epsilon dz \propto u_*^2 U_g + \int_0^h B dz$. Such a linear model of the integral dissipation is obviously a simplification. However, it includes the main terms in question (ZM96). With this estimate of the integral TKE dissipation, the principal balance in Eq. (4) appears to be the balance between the TKE shear production and its buoyancy destruction.

Notice a close analogy between modelling TKE dissipation and modelling pressure terms in the second-moment equations. For example, a widely used linear model for the rapid part of the pressure-gradient-scalar covariance in the scalar-flux equation includes the terms proportional to the mean-shear, buoyancy and Coriolis terms in that equation (see e.g. Zeman, 1981; Mironov, 2001).

Consider first the surface-flux-dominated SBL with $N = 0$. Then $F_h = 0$ and the integral buoyancy destruction of TKE, $\int_0^h B dz$, scales with $hB_s = -u_*^3(h/L)$ (see discussion in ZBRSLC02). The buoyancy destruction is equal to

$(1/2)hB_s$ if the vertical buoyancy flux is a linear function of z , which is a good approximation for the boundary layer in a quasi-steady state (Nieuwstadt, 1984). Using this estimate, we obtain

$$h/L \propto U_g/u_*. \tag{5}$$

In the imposed-stability-dominated SBL, the integral buoyancy destruction of TKE scales with $-u_*^2 hN$ (see ZM96). With this estimate, we obtain

$$hN/u_* \propto U_g/u_*. \tag{6}$$

Notice that the energy flux F_h in the imposed-stability-dominated SBL may not be equal to zero. Its consideration does not change the above scaling estimate, however. As shown by Zilitinkevich (2002) and Soomere and Zilitinkevich (2002), F_h scales with $u_*^2 hN$, that is in the same way as the integral buoyancy destruction.

As Eqs (5) and (6) suggest, an estimate of U_g/u_* is required in order to obtain an expression for h . To this end, one needs to consider the SBL momentum budget and to derive the so-called resistance law that relates the geostrophic velocity components to the components of the surface momentum flux.

3. Momentum budget considerations

For a steady-state SBL, the momentum balance equations are

$$-\partial \tau_x / \partial z + f(v - V_g) = 0, \tag{7}$$

$$-\partial \tau_y / \partial z - f(u - U_g) = 0, \tag{8}$$

where $U_g = -(\rho f)^{-1} \partial P_s / \partial y$ and $V_g = (\rho f)^{-1} \partial P_s / \partial x$ are the geostrophic velocity components along the x and y axes (U_g and V_g are depth-constant, as the barotropic SBL case is considered), ρ is the density (within the framework of the Boussinesq approximation, a constant density is used in the expressions for U_g and V_g) and P_s is the surface pressure. The momentum-flux components are taken to be related to the mean velocity components through the down-gradient approximation as $\tau_x = -K_m \partial u / \partial z$ and $\tau_y = -K_m \partial v / \partial z$, with K_m being the kinematic eddy viscosity.

Examine the simplest case of a constant eddy viscosity $K_m = K_*$. The problem is conveniently considered in complex notation using a frame of reference with the x -axis aligned with the surface stress. Recasting the momentum Eqs (7) and (8) in terms of the dimensionless kinematic

momentum flux $\hat{t} = (\tau_x + i\tau_y)/u_*^2$ and the dimensionless height $\hat{z} = z/h_E$, where

$$h_E = (2K_*/|f|)^{1/2} \quad (9)$$

is the Ekman depth scale (Ekman 1905), we obtain

$$\partial^2 \hat{t} / \partial \hat{z}^2 - 2i \operatorname{sign}(f) \hat{t} = 0. \quad (10)$$

The boundary condition at the surface becomes

$$\hat{t} = -1 \quad \text{at } \hat{z} = 0. \quad (11)$$

The boundary condition at the SBL top reads

$$\hat{t} = 0 \quad \text{at } \hat{z} = h/h_E. \quad (12)$$

The solution to Eq. (10) subject to boundary conditions (11) and (12) is

$$\hat{t} = \frac{\sinh \{ [1 + i \operatorname{sign}(f)] (\hat{z} - h/h_E) \}}{\sinh \{ [1 + i \operatorname{sign}(f)] h/h_E \}}.$$

The velocity profile is then found from this solution and the no-slip condition at the surface as

$$\frac{u(z) + iv(z)}{u_*} = -\frac{u_* h_E}{K_*} \int_0^{\hat{z}} \hat{t}(\hat{z}') d\hat{z}'. \quad (13)$$

Taking Eq. (13) at $\hat{z} = h/h_E$ where the velocity components are equal to the geostrophic velocity components yields the resistance law,

$$\frac{U_g + iV_g}{u_*} = \frac{u_*}{(2K_*|f|)^{1/2}} \times \frac{\cosh \{ [1 + i \operatorname{sign}(f)] h/h_E \} - 1}{\sinh \{ [1 + i \operatorname{sign}(f)] h/h_E \}} [1 - i \operatorname{sign}(f)]. \quad (14)$$

Equation (14) serves to determine U_g/u_* . Then Eqs (5) and (6) yield expressions for the depth of the surface-flux-dominated SBL and imposed-stability-dominated SBL, respectively. Notice, however, that the right-hand side (r.h.s.) of Eq. (14) depends on the yet undetermined dimensionless parameter h/h_E , the ratio of the boundary-layer depth to the Ekman depth scale defined through Eq. (9). An assumption about this ratio is required to close the problem. In the next section, we show how different assumptions about h/h_E lead to different formulations for the SBL depth.

One comment is in order concerning the expression for the integral shear production of TKE, $u_*^2 U_g$, which follows from Eq. (14). As a consequence of the use of constant eddy viscosity all the way through the boundary layer, the expression for $u_*^2 U_g$ does not contain a $(u_*^3/\kappa) \ln(h/z_0)$ term, z_0 being the surface aerodynamic roughness. This term would appear in the expression for $u_*^2 U_g$ by virtue of the surface-layer similarity, which states that in the vicinity of the surface (more specifically, at $z/\min(h, L) \ll 1$) the (negative of) momentum flux is approximately equal to u_*^2 and the velocity gradient behaves as $u_*/(\kappa z)$. However, the $(u_*^3/\kappa) \ln(h/z_0)$ term that enters the TKE budget through the integral shear production is exactly cancelled by the term of opposite sign that enters the budget through the integral dissipation. Indeed, close to the surface the TKE dissipation behaves, to leading order, as $u_*^3/(\kappa z)$. The integration of this expression over the boundary layer gives $(u_*^3/\kappa) \ln(h/z_0)$. Hence, the term with $\ln(h/z_0)$ is not present in the final expressions for h (cf. ZM96).

4. Alternative formulations for the SBL depth revisited

Now we show that the Zilitinkevich (1972) and Pollard *et al.* (1973) scales on the one hand, and the Kitaigorodskii (1960) and Kitaigorodskii and Joffre (1988) scales on the other hand, follow from the consideration of the TKE and momentum budgets, depending upon which assumption is made about the ratio h/h_E of the SBL depth to the Ekman depth. Recall that the four SBL depth scales in Table I hold when the static stability due to the surface buoyancy flux or due to the stable density stratification at the boundary-layer outer edge is sufficiently strong. This property is expressed quantitatively in terms of the dimensionless stability parameters appropriate for the two SBL regimes as follows:

$$L|f|/u_* \ll 1, \quad |f|/N \ll 1. \quad (15)$$

Putting it differently, L or u_*/N , respectively, should be small compared with the Rossby and Montgomery (1935) scale $u_*/|f|$ pertinent to the truly neutral boundary layer. In fact, Eq. (15) specifies the limits of applicability of the expressions for the SBL depth. It states that they are valid asymptotically as the stability is strong. In order to cover the entire range of stability conditions, from strongly stable to truly neutral, an interpolation formula is required that turns into $h \propto u_*/|f|$ as B_s and N tend to zero.

- (a) Assume that h is of the order of h_E . Equation (14) yields $U_g/u_* \propto u_*/(K_*|f|)^{1/2}$. Eliminating U_g/u_* from this expression, from the expression $h \propto (K_*/|f|)^{1/2}$ that holds by virtue of the assumption about h/h_E and from Eq. (5), then solving for K_* and h , we obtain $K_* \propto u_* L$ and $h \propto (u_* L/|f|)^{1/2}$. The last expression is recognized as the Zilitinkevich (1972) formulation for the depth of the surface-flux-dominated SBL. Actually, it is the very assumption that $h/h_E = \text{constant}$ that makes it possible to arrive at this estimate. Using Eq. (6) in lieu of Eq. (5), we obtain $K_* \propto u_*^2/N$ and $h \propto u_*/|Nf|^{1/2}$. The latter expression is recognized as the Pollard *et al.* (1973) formulation for the depth of the imposed-stability-dominated SBL. Again, it is the assumption that $h/h_E = \text{constant}$ that enables one to obtain this result.
- (b) Assume, as distinct from (a), that $h \ll h_E$. Expanding the r.h.s. of Eq. (14) into a power series in h/h_E and keeping only the leading-order term, we obtain $U_g/u_* \propto hu_*/K_*$. This expression, along with Eq. (5), yields an estimate of the eddy viscosity, $K_* \propto u_* L$, and the inequality $(h/L) (L|f|/u_*)^{1/2} \ll 1$ that serves to determine h . Since the dimensionless parameter $L|f|/u_*$ is small compared with 1, as stated by Eq. (15), the inequality may be satisfied in many different ways including $h/L = \text{constant}$. This expression is recognized as the Kitaigorodskii (1960) formulation for the depth of the surface-flux-dominated SBL. Using Eq. (6) in lieu of Eq. (5), we obtain $K_* \propto u_*^2/N$ and $(hN/u_*) (|f|/N)^{1/2} \ll 1$. Given the second member of Eq. (15), the last inequality may be satisfied in many different ways including $hN/u_* = \text{constant}$. This expression is recognized as the Kitaigorodskii and Joffre (1988) formulation for the depth of the imposed-stability-dominated SBL.

Thus, both sets of alternative SBL-depth formulations, namely the formulations that include f and the formulations that do not, prove to be consistent with the TKE-budget and momentum-budget considerations. However, the assumptions invoked to derive these formulations are different. A critical issue is the assumed (postulated) ratio h/h_E of the SBL depth to the Ekman depth. Taking $h/h_E = \text{constant}$ (assumption (a)) produces SBL-depth expressions that incorporate the Coriolis parameter however strong the static stability might be. Taking $h/h_E \ll 1$ (assumption (b)) may eliminate the Coriolis parameter from the expressions for the equilibrium SBL depth. In view of an approximate character of the theory, it seems difficult to give preference to (a) over (b), or vice versa. Hence, the scales $(u_*L/|f|)^{1/2}$ and L for the surface-flux-dominated SBL and $u_*/|Nf|^{1/2}$ and N/u_* for the imposed-stability-dominated SBL should be considered equally justified. In the next section, we propose somewhat more general SBL-depth formulations that incorporate these scales as particular cases.

5. Generalized formulations

Consider the following power-law expression pertinent to the surface-flux-dominated SBL:

$$h/L \propto (L|f|/u_*)^{-\gamma}, \tag{16}$$

where the exponent γ lies in the range from 0 to 1/2. Clearly, the Zilitinkevich (1972) formulation, $h \propto (u_*L/|f|)^{1/2}$, is recovered with $\gamma = 1/2$. It is straightforward to verify that the inequality $(h/L)(L|f|/u_*)^{1/2} \ll 1$ is satisfied with any value of $0 \leq \gamma < 1/2$, provided that the dimensionless stability parameter $L|f|/u_*$ is small (see discussion in the previous section concerning the asymptotic nature of the SBL depth formulations). With $\gamma = 0$, the equilibrium SBL depth ceases to depend on the Coriolis parameter in its explicit form and Eq. (16) turns into the Kitaigorodskii (1960) formulation $h \propto L$. In this way, a power-law formulation (16) is a generalization of the two previous formulations, as it incorporates the two alternative depth scales for the surface-flux-dominated SBL as particular cases.

For the imposed-stability-dominated SBL, a generalized power-law expression reads

$$hN/u_* \propto (|f|/N)^{-\delta}, \tag{17}$$

where $0 \leq \delta \leq 1/2$. The Pollard *et al.* (1973) formulation, $h \propto u_*/|Nf|^{1/2}$, is recovered with $\delta = 1/2$. The Kitaigorodskii and Joffe (1988) formulation, $h \propto u_*/N$, which does not incorporate the Coriolis parameter, is recovered with $\delta = 0$.

The power-law expressions (16) and (17) may be viewed as examples of so-called self-similarity of the second kind. In order to elucidate this feature, consider the following example. Suppose a physical variable in question depends on three governing parameters. If these governing parameters have two independent dimensions, the Buckingham (1914) Π theorem states that the relation sought can be written in dimensionless form as

$$\hat{v} = F(\Pi). \tag{18}$$

Here, \hat{v} is the variable in question made dimensionless with an appropriate scale, and Π is a single dimensionless

parameter that can be composed of three dimensional governing parameters with two independent dimensions. We are interested in the limiting behaviour of the function F as the dimensionless parameter Π tends to zero. There are two essentially different possibilities (Barenblatt, 1982, 1996).

- (i) As $\Pi \rightarrow 0$, F tends to a constant different from zero. This case is referred to as self-similarity of the first kind or complete self-similarity with respect to the parameter Π .
- (ii) As $\Pi \rightarrow 0$, F does not have a finite non-zero limit, however, the following power-law asymptotics holds true:

$$\hat{v} = C\Pi^\alpha, \tag{19}$$

where C is a dimensionless constant. That is, the variable \hat{v} depends on the dimensionless parameter Π no matter how small this parameter may be. This case is referred to as self-similarity of the second kind or incomplete self-similarity with respect to the parameter Π . Importantly, the exponent α cannot be determined by dimensional analysis.

Actually, there is a third possibility where neither (i) nor (ii) takes place. In that case there is no self-similarity with respect to the parameter Π .

Self-similarity of the second kind is often encountered in physics and mathematics. A large number of illustrative examples from different areas of study, ranging from fractal curves to the theory of elasticity and including geophysical fluid dynamics, are considered by Barenblatt (1982, 1996). The atmospheric surface-layer flux-profile relationships as viewed from the self-similarity standpoint are discussed in Kramm and Herbert (2009). A close analogy between the self-similarity of the second kind and the renormalization group theory that found some use in turbulence studies is discussed in Goldenfeld *et al.* (1989). Self-similar solutions of the second kind occur in numerous physical problems as so-called intermediate asymptotics. An intermediate asymptotic is an asymptotic solution to the problem that holds at $X_1 \ll x \ll X_2$, where X_1 and X_2 are governing parameters that have the same physical dimensions as an independent variable x (see Barenblatt, 1982, 1996). In statistical physics, this is referred to as scaling (Goldenfeld *et al.*, 1989).

We may now consider Eq. (16) in terms of self-similarity of the second kind. The dependence of h on the parameters governing the surface-flux-dominated SBL, namely L , f and u_* , may be represented in dimensionless form as $h/L = F(L|f|/u_*)$, cf. Eq. (18). The parameter $L|f|/u_*$ is the only dimensionless parameter that can be composed from L , f and u_* , since these three quantities have two independent dimensions. A close analogy between the power laws (16) and (19) is immediately seen, suggesting that Eq. (16) is a self-similar expression of the second kind. Notice that according to the Barenblatt (1982, 1996) classification, the case $\gamma = 0$ should be viewed as an example of self-similarity of the first kind, or complete self-similarity with respect to the parameter $L|f|/u_*$. It is interesting to note further that the validity of Eq. (16) with $\gamma > 0$ implies the validity of the following double inequality:

$$L \ll h \ll u_*/|f|. \tag{20}$$

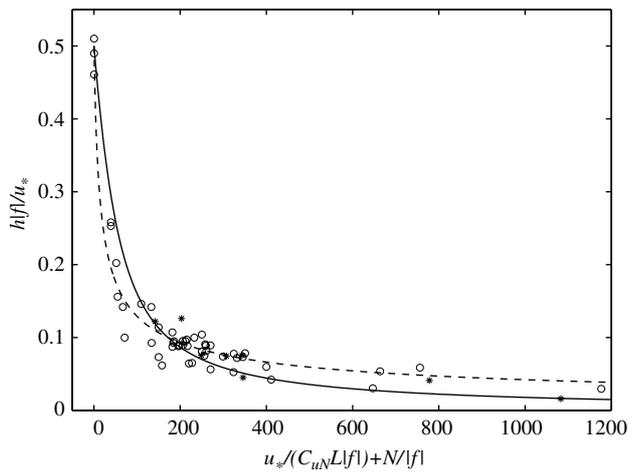


Figure 1. Dimensionless SBL depth $h|f|/u_*$ as a function of the composite stability parameter $u_*/(C_{uN}L|f|) + N/|f|$. The solid curve shows the Zilitinkevich and Mironov (1996) formulation, Eq. (23), with $C_n = 0.5$, $C_s = 10$ and $C_i = 18$. The dashed curve shows the Zilitinkevich *et al.* (2002) formulation, Eq. (22), with $C_R = 0.5$, $C_s = 1$ and $C_{uN} = 0.56$. Asterisks are data from measurements in the atmospheric and benthic SBLs. Circles are LES data. See text for details.

Although Eq. (20) holds for the quantity sought for rather than for an independent variable (see above), it suggests that the SBL depth formulation (16) has an intermediate asymptotic character at $\gamma > 0$.

A power-law expression (17) for the depth of the imposed-stability-dominated SBL can also be considered in terms of self-similarity of the second kind. Indeed, the dependence of h on the parameters governing the imposed-stability-dominated SBL may be represented in dimensionless form as $hN/u_* = F(|f|/N)$, since the parameter $|f|/N$ is the only dimensionless parameter that can be composed from N , f and u_* . A close analogy between the power laws (17) and (19) suggests that Eq. (17) is a self-similar solution of the second kind. The case $\delta = 0$ should be viewed as an example of self-similarity of the first kind, or complete self-similarity with respect to the parameter $|f|/N$. The validity of Eq. (17) with $\delta > 0$ implies the validity of the following double inequality:

$$u_*/N \ll h \ll u_*/|f|, \quad (21)$$

which suggests that the SBL depth formulation (17) has an intermediate asymptotic character at $\delta > 0$.

Recall that γ and δ cannot be determined by dimensional analysis. These exponents should be determined either from an exact solution to equations governing the structure of mean fields and turbulence in the SBL, or, if such a solution is not known, from experimental data.

6. Data

In Figure 1, the SBL depth evaluated from Eq. (1) in its reduced form, without the intermediate scales, and from Eq. (2) is compared with data from measurements in natural conditions and from large-eddy simulations. Observational data include the point from Weatherly and Martin (1978), representing the benthic boundary layer, the atmospheric boundary-layer data from Lenschow *et al.* (1988a, 1988b, Flights No. 5 and 6) and the atmospheric boundary-layer data from Overland and Davidson (1992, the median and upper hinge estimates for northerly winds

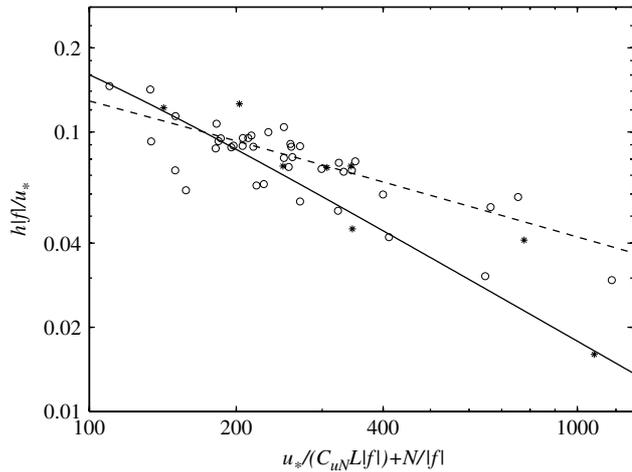


Figure 2. As Figure 1 but on a log-log scale for values of the composite stability parameter $u_*/(C_{uN}L|f|) + N/|f|$ in excess of 100.

and the lower hinge, median and upper hinge estimates for southerly winds). The LES data are from Mason and Thomson (1987, case B10), Mason and Derbyshire (1990, case B), Andr n and Moeng (1993), Brown *et al.* (1994, cases BA10, BA10HR, BC10(12k6) and BC10(28k8)), Andr n (1995, case SGSM2), Saiki *et al.* (2000), Kosovi c and Curry (2000, NL cases generated with the nonlinear subgrid-scale model), Zilitinkevich and Esau (2003, barotropic cases TrNBt through SBT3) and Beare and MacVean (2004, cases C91 and F93).

In order to show all data points on the same plot, the ZBRSLC02 Eq. (2) is rearranged to give

$$\left(\frac{h|f|}{u_*}\right) \left[\frac{1}{C_R^2} + \frac{C_{uN}}{C_s^2} \left(\frac{1}{C_{uN}} \frac{u_*}{|f|L} + \frac{N}{|f|} \right) \right]^{1/2} = 1, \quad (22)$$

with the estimates $C_R = 0.5$, $C_s = 1$ and $C_{uN} = 0.56$ given by Zilitinkevich and Esau (2003). Rearranging the ZM96 Eq. (1), we obtain

$$\frac{1}{C_n^2} \left(\frac{h|f|}{u_*}\right)^2 + \frac{1}{C_i} \left(\frac{C_i}{C_s} \frac{u_*}{|f|L} + \frac{N}{|f|}\right) \left(\frac{h|f|}{u_*}\right) = 1, \quad (23)$$

with the estimates $C_n = 0.5$, $C_s = 10$ and $C_i = 20$ given by ZM96. In the case $C_{uN} = C_s/C_i$, both theoretical curves and all data points can be shown on the same plot in terms of the dimensionless SBL depth $h|f|/u_*$ versus the composite stability parameter $u_*/(C_{uN}L|f|) + N/|f|$. To do so, we adjust the constant C_i in Eq. (23) so that $C_i = C_s/C_{uN}$. Using $C_{uN} = 0.56$ and $C_s = 10$, we obtain $C_i = 17.86 \approx 18$, which is very close to the ZM96 estimate of $C_i = 20$ and is within the accuracy of the analysis.

As Figure 1 suggests, both the ZBRSLC02 formulation and the ZM96 formulation show a satisfactory agreement with data. On average, Eq. (22) reveals a somewhat higher bias than Eq. (23), 1.8×10^{-3} versus 1.1×10^{-3} , but a somewhat lower root-mean-square (r.m.s.) error, 2.1×10^{-2} versus 3.0×10^{-2} , respectively. The corresponding dimensional values of bias and r.m.s. error are 15.3 m and 81.0 m, respectively, for Eq. (22) and 8.3 m and 99.7 m, respectively, for Eq. (23).

Figure 2 is similar to Figure 1, except that the two theoretical curves and the data are shown for values of the composite stability parameter $u_*/(C_{uN}L|f|) + N/|f|$

in excess of 100. Notice that in the case of sufficiently strong stability, Eq. (22) corresponds to $\gamma = \delta = 1/2$ (upper limit for the exponents) in the generalized SBL-depth formulations (16) and (17) and Eq. (23) corresponds to $\gamma = \delta = 0$ (lower limit for the exponents). As Figure 2 suggests, some data points support Eq. (22) whereas other points support Eq. (23). It is, therefore, hardly possible to decide between the two formulations for the equilibrium SBL depth on purely empirical grounds. The data scatter is large, rendering it impossible to determine γ and δ with a fair degree of accuracy. It should be noted that the dependences shown in Figures 1 and 2 may be subject to self-correlation (Klipp and Mahrt, 2004); f appears in the numerator of $h|f|/u_*$ and in the denominator of $u_*/(C_{uN}L|f|) + N/|f|$. An attempt to plot dimensional SBL depth h versus the composite stability parameter $u_*/(C_{uN}L|f|) + N/|f|$ (not shown) does not reduce the data scatter. Thus, our conclusion concerning the impossibility of discriminating between the alternative formulations on the basis of available data remains in force.

As regards practical applications, both multi-limit formulations (22) and (23) or similar formulations with γ and δ in the range from 0 to 1/2 can be used, keeping in mind their inherent uncertainties. The multi-limit formulations are expected to give similar results for stability conditions typical of the atmospheric and oceanic SBLs, provided the disposable dimensionless coefficients are appropriately tuned.

7. Conclusions

The effect of rotation on the equilibrium depth of a stably stratified barotropic boundary layer is analyzed. Two alternative depth scales have been proposed for a SBL dominated by surface buoyancy flux. Kitaigorodskii (1960) assumed that the Earth's rotation is no longer important as static stability becomes strong and that the SBL depth scales with the Obukhov length. Zilitinkevich (1972) proposed an alternative scale that depends on the Coriolis parameter no matter how strong the static stability. Similarly, two alternative depth scales have been proposed for an SBL dominated by static stability at its outer edge. The depth scale introduced by Kitaigorodskii and Joffre (1988) does not depend on the Coriolis parameter, whereas the Pollard *et al.* (1973) scale does.

The analysis in sections 2 and 3 suggests that all the above depth-scale formulations are consistent with TKE-budget and momentum-budget considerations. The assumptions invoked to derive alternative formulations are different, however. A critical issue is the ratio h/h_E of the SBL depth to the Ekman depth. Assuming h/h_E to be constant yields SBL depth expressions that incorporate the Coriolis parameter, however strong static stability might be. Assuming h/h_E to be small (as compared with 1) allows elimination of the Coriolis parameter from the expressions for the equilibrium SBL depth.

It is demonstrated that in the case of sufficiently strong static stability the alternative depth-scale formulations represent particular cases of more general power-law formulations, Eqs (16) and (17), with the power-law exponents γ and δ in the range from 0 to 1/2. With $\gamma=1/2$ and $\delta=1/2$, Eqs (16) and (17) yield the Zilitinkevich (1972) scale and the Pollard *et al.* (1973) scale, respectively. In the limit $\gamma=0$ and $\delta=0$, the SBL depth scales cease

to depend on the Coriolis parameter in their explicit form and the formulations proposed by Kitaigorodskii (1960) and Kitaigorodskii and Joffre (1988), respectively, are recovered. Interpretation of the generalized power-law formulations is proposed in terms of self-similarity of the second kind (also referred to as incomplete self-similarity). Self-similarity of the second kind places the generalized scaling formulations (16) and (17) into a well-established physical and mathematical framework. Furthermore, the notion of intermediate asymptotics helps to clarify the limits of applicability of the proposed generalized relations through the double inequalities (20) and (21).

The power-law exponents in Eqs (16) and (17) cannot be determined by dimensional analysis. To do this would require an exact solution to equations governing the structure of mean fields and turbulence in the SBL. Since such a solution is not known, the exponents should be evaluated from experimental data. Available observational and numerical data are uncertain. They render it impossible to evaluate γ and δ to sufficient accuracy and to decide conclusively between the alternative formulations for the SBL depth. As regards practical applications, multi-limit formulations, e.g. Eqs (22) and (23) or similar formulations with power-law exponents in the range from 0 to 1/2, are expected to give similar results for stability conditions typical of the atmospheric and oceanic SBLs, provided the disposable dimensionless coefficients are appropriately tuned.

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