J. Fluid Mech. (2004), *vol.* 498, *pp.* 333–352.
DOI: 10.1017/S0022112003006803 Printed in the United Kingdom

Unsteady convectively driven flow along a vertical plate immersed in a stably stratified fluid

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(Received 19 September 2002 and in revised form 25 August 2003)

This paper revisits the classical problem of convectively driven one-dimensional (parallel) flow along an infinite vertical plate. We consider flows induced by an impulsive (step) change in plate temperature, a sudden application of a plate heat flux, and arbitrary temporal variations in plate temperature or plate heat flux. Provision is made for pressure work and vertical temperature advection in the thermodynamic energy equation, processes that are generally neglected in previous one-dimensional studies of this problem. The pressure work term by itself provides a relatively minor refinement of the Boussinesq model, but can be conveniently combined with the vertical temperature advection term to form a single term for potential temperature advection. Vertical motion of air in a statically stable environment (stable potential temperature stratification) is associated with a simple negative feedback mechanism: warm air rises, expands and cools relative to the environment, whereas cool air subsides, compresses and warms relative to the environment. Exact solutions of the viscous equations of motion are obtained by the method of Laplace transforms for the case where the Prandtl number is unity. Pressure work and vertical temperature advection are found to have a significant impact on the structure of the solutions at later times.

1. Introduction

The transient natural convection flow of a viscous fluid adjacent to vertical surfaces is a fundamental problem in fluid mechanics and heat transfer, with significance for a variety of engineering applications (Gebhart *et al.* 1988). The simplest form of this problem is one-dimensional transient convective flow adjacent to an infinite vertical plate, first considered by Illingworth (1950) for an impulsive change in plate temperature. Siegel (1958), Menold & Yang (1962), Schetz & Eichhorn (1962), Goldstein & Briggs (1964), and Das, Deka & Soundalgekar (1999) have obtained analytic solutions to this problem for a variety of temporal variations in plate temperature and plate heat flux. In these studies, pressure work is neglected and ambient thermal stratification is not considered. Accordingly, the thermodynamic energy equation reduces to the standard one-dimensional heat conduction equation. After solving this equation for the temperature field, the vertical velocity is recovered from the vertical equation of motion which has the form of a diffusion equation with inhomogeneous buoyancy forcing term. These exact unsteady solutions of the Boussinesq equations are potentially valuable as simple conceptual/pedagogical models of natural convection as well as tools for validating numerical models of convection.

It has been suggested that these solutions can also be applied to the more realistic scenario of convectively driven flow adjacent to a semi-infinite vertical plate (plate with a lower edge) in regions where the disturbance originating near the lower plate edge has not yet propagated (Gebhart *et al.* 1988). The passage of this 'leading-edge effect' heralds the breakdown of the one-dimensional regime and the onset of two-dimensionality (transient two-dimensional flow and eventual two-dimensional steady state). Regions further above the leading edge remain in the one-dimensional regime for longer periods of time before passing into the two-dimensional regime. A number of theoretical studies (e.g. Goldstein & Briggs 1964; Brown & Riley 1973; Ingham 1985) have sought to explain the propagation of the leading-edge signal as an advective effect. However, the propagation speed retrieved from the maximum boundary-layer advection speed was found to be generally lower than the speeds observed in laboratory experiments (e.g. Mahajan & Gebhart 1978; Joshi & Gebhart 1987). One- and two-dimensional solutions, the leading-edge effect, and other aspects of transient natural convection adjacent to vertical surfaces are reviewed in Gebhart *et al.* (1988). More recent studies (Armfield & Patterson 1992; Daniels & Patterson 1997, 2001; Patterson *et al.* 2002) suggest that the leading-edge effect is associated with the propagation of waves excited at the plate edge by the impulsive nature of the start-up procedure.

However, the paradigm of a flow that remains one-dimensional until its disruption by the leading-edge effect must be modified in the case of flow instability. In some of the laboratory experiments of flows driven by sudden application of a plate heat flux (Joshi & Gebhart 1987) and impulsive change of plate temperature (Brooker *et al.* 2000; Patterson *et al.* 2002), the evolving one-dimensional flows became unstable prior to the arrival of the leading-edge signal. The boundary-layer instabilities were probably due to amplification of disturbances generated by the impulsive nature of the start-up procedure or by vibrations of the experimental apparatus.

The present study refines the classical theory of one-dimensional transient convectively driven flow along a vertical plate by including the pressure work term in the thermodynamic energy equation. It also extends the classical theory by making provision for a linearly varying ambient temperature. As we will see, in the context of the one-dimensional model, the pressure work and vertical temperature advection terms are of the same form so the refinement and extension can be effected simultaneously by combining both processes into a single advection term. With attention restricted to a perfect gas with a Prandtl number of unity, solutions are readily obtained by the method of Laplace transforms. Provision for temperature stratification or pressure work allows the unsteady solution to approach a steady state at large times, whereas if there is no temperature stratification or pressure work (hereinafter referred to as classical solution), the solution grows without bound. As in the classical case, the new solutions will only be appropriate for times prior to the arrival of the leading-edge effect and prior to the onset of any flow instabilities. The steady-state solution for the stratified fluid case has already been obtained by Gill (1966), and its linear stability has been analysed by Gill & Davey (1969) and Bergholz (1978). In those studies, the stability was found to decrease with increasing plate perturbation temperature and increase with increasing stratification. In light of those studies and the experiments of Joshi & Gebhart (1987), Brooker *et al.* 2000 and Patterson *et al.* 2002, we anticipate that the main interest in our solutions will be in cases where the temperature stratification is large enough to delay (or prevent) flow instability. For weak temperature stratifications, the deviations of our solutions from the corresponding classical solutions may not become apparent before the flow becomes unstable.

Exact unsteady solutions of the viscous flow equations for one-dimensional natural convection with provision for an ambient stratification are apparently limited to the studies of Park & Hyun (1998) and Park (2001). Those investigations were concerned with transient natural convection in a vertical channel whose two sidewalls were subjected to an impulsive (step) change in temperature. The solution was obtained for arbitrary Prandtl number with an eigenfunction expansion method. For large times, this transient solution approached the steady-state channel solution obtained by Elder (1965). As in the single-plate problem, stable stratification was found to stabilize the steady-state solution (Bergholz 1978), except for values of the stratification parameter near the transition region between stationary and travelling-wave instabilities. The single-plate transient solutions obtained in our present study complement the double-plate (channel) transient solution of Park & Hyun (1998) and Park (2001). Although we consider a greater variety of plate thermal forcings (impulsive changes in plate perturbation temperature and heat flux as well as arbitrary temporal changes in plate temperature and heat flux), our analysis is restricted to a Prandtl number of unity.

The outline of the paper is as follows. In § 2, we formulate the problem of one-dimensional natural convection for a fluid whose thermal expansion coefficient is that of a perfect gas. The governing equations are introduced and reduced to a single fourth-order linear partial differential equation for the perturbation temperature. This equation is solved in §§ 3 and 4 by the method of Laplace transforms for the case where the Prandtl number is unity. The case of an impulsively changed plate perturbation temperature is treated in § 3, and the case of a suddenly applied plate heat flux is treated in § 4. In § 5, the new solutions are compared to the classical solutions in which pressure work is neglected and the environment is considered to be isothermal. In § 6, we present solutions for flows driven by arbitrary temporal variations in plate perturbation temperature or plate heat flux.

2. Governing equations

Consider a Cartesian coordinate system in which the z-axis opposes the gravity vector, the (y, z)-plane coincides with an infinite vertical plate, the x-axis is directed perpendicular to the plate, and fluid fills the region $x \geqslant 0$. The fluid is quiescent with zero horizontal temperature gradient until thermal conditions at the plate are abruptly changed at $t = 0$. The ensuing motion is one-dimensional with the only non-zero velocity component, the vertical velocity w, varying only in the x-direction. Accordingly, the mass conservation equation (incompressibility condition) is trivially satisfied. In order for the horizontal equations of motion to be satisfied (albeit trivially), the horizontal pressure gradient force must be zero everywhere. Thus $\partial p/\partial x = 0$, and the local pressure $p(x, z, t)$ must equal the pressure at $x \to \infty$, which is the environmental pressure $p_\infty(z)$. Since there is no motion or thermal disturbance far from the plate, p_∞ satisfies the hydrostatic equation, $\mathrm{d}p_\infty/\mathrm{d}z = -\rho_\infty g$, where $\rho_\infty(z)$ is the environmental density. Accordingly, p itself satisfies the hydrostatic equation based on the environmental density, $\mathrm{d}p/\mathrm{d}z = -\rho_\infty g$.

With the density decomposed into its environmental and perturbation components, $\rho(x, z, t) = \rho_\infty(z) + \rho'(x, t)$, the Boussinesq form of the vertical equation of motion is

$$\frac{\partial w}{\partial t} = -g\frac{\rho'}{\rho_r} + \nu\frac{\partial^2 w}{\partial x^2}. \tag{1}$$

Here, the subscript 'r' denotes a constant reference value, and ν is a constant kinematic viscosity coefficient. We can also decompose the temperature into its environmental and perturbation components, $T(x, z, t) = T_\infty(z) + T'(x, t)$. The Boussinesq (linearized) form of the equation of state, $\rho' = -(\rho_r/T_r)T'$, allows us to eliminate ρ' in (1) in favour of T':

$$\frac{\partial w}{\partial t} = g\frac{T'}{T_r} + \nu\frac{\partial^2 w}{\partial x^2}. \tag{2}$$

The term gT'/T_r is the buoyancy force per unit mass of the fluid.

Now consider the thermodynamic energy equation for a perfect gas of constant thermal conductivity (Schlichting 1979),

$$\rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} + k\frac{\partial^2 T}{\partial x^2}. \tag{3}$$

Here, $\mathrm{D}/\mathrm{D}t = \partial/\partial t + w\partial/\partial z$ is the one-dimensional total derivative operator, c_p is the specific heat at constant pressure and k is the thermal conductivity. The value of unity for the coefficient of the pressure work term $\mathrm{D}p/\mathrm{D}t$ is appropriate for a perfect gas in which the thermal expansion coefficient varies inversely with temperature. We have neglected the viscous dissipation term which is generally smaller than the other terms in (3), including the pressure work term (Ackroyd 1974; Napolitano, Carlomagno & Vigo 1977; Gebhart *et al.* 1988; Mahajan & Gebhart 1989). The pressure work term itself is small and is neglected in the conventional Boussinesq approximation (Kundu & Cohen 2002), the error being of the same order of magnitude as errors due to the neglect of variations in the material properties κ and ν (Ackroyd 1974). Our retention of the pressure work term amounts to a slight refinement of the Boussinesq model. In meteorology, the pressure work and total derivative of temperature terms are often combined into a single term, $\rho c_p \mathrm{D}T/\mathrm{D}t - \mathrm{D}p/Dt = \rho c_p T \mathrm{D}\ln\Theta/\mathrm{D}t$, where $\Theta \equiv T(p_s/p)^{R_d/c_p}$ is the potential temperature, R_d is the gas constant of the dry air, and p_s is a reference pressure level. The potential temperature is the temperature a parcel of dry air with temperature T and pressure p would have it was brought to the reference pressure level by an adiabatic process (Holton 1992). Pressure work will be found to have only a small impact on the solution except in the special case of zero temperature stratification, in which case pressure work enables the solution to approach a steady state at large times (its impact at small times being negligible). However, in the absence of temperature stratification, a flow would probably become unstable for all but the smallest thermal forcings before pressure work effects became apparent. Thus, the main practical interest of our study is likely to be in cases where ambient stratification is large enough to prevent or delay instability, in which case the temperature advection term $w\ \partial T/\partial z$ would probably dominate the pressure work term. (We note further that if the pressure work term is omitted, the resulting system of equations should be equally applicable to Boussinesq flow of liquids or gases.)

Since $T(x,z,t) = T_\infty(z) + T'(x,t)$ and $p(x,z,t) = p_\infty(z)$, where $\mathrm{d}p_\infty/\mathrm{d}z = -\rho_\infty g$, (3) becomes

$$\frac{\partial T'}{\partial t} = -\frac{\mathrm{d}T_\infty}{\mathrm{d}z}w - \frac{\rho_\infty g}{\rho c_p}w + \frac{k}{\rho c_p}\frac{\partial^2 T'}{\partial x^2}. \tag{4}$$

Restricting attention to linearly varying environmental temperatures $T_\infty(z)$, approximating ρ_∞/ρ as unity, and treating the thermal diffusivity $\kappa \equiv k/(\rho c_p)$ as constant, (4) becomes

$$\frac{\partial T'}{\partial t} = -\gamma w + \kappa\frac{\partial^2 T'}{\partial x^2}, \tag{5}$$

where $\gamma \equiv \mathrm{d}T_\infty/\mathrm{d}z + g/c_p$ is a constant parameter.

The $-\gamma w$ term in (5) arising from the combined effects of pressure work and vertical temperature advection introduces a coupling between w and T' beyond the appearance of the buoyancy force in the vertical equation of motion, (2). Since the temperature gradient in a statically neutral adiabatic environment is $\mathrm{d}T/\mathrm{d}z|_{ad} = -g/c_p$ (Holton 1992), we can interpret γ as the difference between the environmental temperature gradient and the temperature gradient in a statically neutral adiabatic environment. The value of γ is proportional to the gradient of the environmental potential temperature, $\mathrm{d}\Theta_\infty/\mathrm{d}z = (\Theta_\infty/T_\infty)/(\mathrm{d}T_\infty/\mathrm{d}z + g/c_p)$, and thus when we refer to γ as a stratification parameter, we are referring to potential temperature stratification. The environment is statically stable if $\gamma > 0$, statically neutral if $\gamma = 0$, and statically unstable if $\gamma < 0$. Under statically stable conditions ($\gamma > 0$) the $-\gamma w$ term provides a simple negative feedback in (1), (2) and (5): warm fluid rises, expands and cools relative to the environment, whereas cool fluid subsides, compresses and warms relative to the environment. Provision for this feedback adds a new level of realism to the classical problem.

The plate boundary conditions for $t > 0$ are the no-slip condition, $w(0, t) = 0$, and either a specified constant temperature perturbation or a specified constant kinematic heat flux (later we will consider arbitrary temporal variations of plate perturbation temperature and plate heat flux). The vertical velocity and perturbation temperature fields are assumed to vanish far from the plate.

We non-dimensionalize variables with the intent of clearing the governing equations and boundary conditions of all parameters except the Prandtl number. The independent variables x and t are non-dimensionalized as

$$\xi \equiv x\frac{(g\gamma/T_r)^{1/4}}{\sqrt{\nu}}, \quad \tau \equiv t\left(\frac{g\gamma}{T_r}\right)^{1/2}. \tag{6}$$

The non-dimensionalization of the dependent variables T' and w depends on conditions at the plate. If the plate perturbation temperature is constant, $T'(0, t) = T'_0$, we use

$$\theta \equiv \frac{T'}{T'_0}, \quad W \equiv \frac{w}{T'_0}(\gamma T_r/g)^{1/2}. \tag{7a}$$

However, if the heat flux is constant, $Q = -k\partial T'/\partial x(0, t)$, we use

$$\theta \equiv T'\frac{k(g\gamma/T_r)^{1/4}}{Q\sqrt{\nu}}, \quad W \equiv w\frac{k\gamma^{3/4}(T_r/g)^{1/4}}{Q\sqrt{\nu}}. \tag{7b}$$

In either case, the vertical equation of motion, (2), and thermodynamic energy equation, (5), become

$$\frac{\partial W}{\partial \tau} = \theta + \frac{\partial^2 W}{\partial \xi^2}, \tag{8}$$

$$\frac{\partial \theta}{\partial \tau} = -W + \frac{1}{Pr}\frac{\partial^2 \theta}{\partial \xi^2}, \tag{9}$$

where $Pr \equiv \nu/\kappa$ is the Prandtl number. The non-dimensional boundary conditions are

$$W(0, \tau) = 0, \quad W(\infty, \tau) = 0, \quad \theta(\infty, \tau) = 0, \tag{10}$$

and either

$$\theta(0, \tau) = 1 \text{ (constant perturbation temperature)}, \tag{11a}$$

or

$$\frac{\partial \theta}{\partial \xi}(0,\tau) = -1 \text{ (constant heat flux).} \tag{11b}$$

Using (9) to eliminate W from (8), we obtain a fourth-order linear partial differential equation for θ,

$$\frac{\partial^2 \theta}{\partial \tau^2} + \frac{1}{Pr}\frac{\partial^4 \theta}{\partial \xi^4} - \left(1 + \frac{1}{Pr}\right)\frac{\partial^3 \theta}{\partial \xi^2 \partial \tau} + \theta = 0. \tag{12}$$

In the following sections, we solve (12) by the method of Laplace transforms for the analytically tractable case where $Pr = 1$. For the case of arbitrary Pr, the Laplace transform technique would lead to a difficult inverse transformation step (integrand of the Bromwich integral would be a complicated multivalued function). This problem will be the subject of a future investigation.

Our non-dimensionalization for the constant plate perturbation temperature and constant plate heat flux problems has cleared the governing equations and boundary conditions of all parameters (except Pr). Accordingly, the solution curves for θ and W for these problems will be universal in the sense that they only need to be computed once. Any dimensional solution can be obtained from these curves by reverting back to dimensional variables. In particular, the non-dimensionalization (6) is such that when we revert to dimensional distances $x \propto \xi\gamma^{-1/4}$ and times $t \propto \tau\gamma^{-1/2}$, the solution curves for θ and W become progressively stretched along the x- and t-axes as the stratification parameter γ decreases. Thus, locations of maxima and minima (e.g. peak vertical velocity), are at further distances from the plate and occur at later times as γ decreases. Moreover, since (6) does not involve any plate forcing parameter (temperature perturbation T_0' or heat flux Q), all space and time scales characterizing the flow are independent of these forcing parameters. However, in view of (7*a*) and (7*b*), the dimensional vertical velocity and perturbation temperature fields vary linearly with these parameters. The dimensional vertical velocity decreases with increasing stratification parameter: $w \propto \gamma^{-1/2}$ in the case of constant plate perturbation temperature, while $w \propto \gamma^{-3/4}$ in the case of constant plate heat flux.

3. Impulsive (step) change in plate temperature

3.1. *Solution by Laplace transforms*

Consider the case when the plate temperature perturbation is impulsively changed to a non-zero value. Multiplying (12) by $e^{-s\tau}$ and integrating from $\tau = 0$ to $\tau = \infty$ (integrating by parts wherever possible) yields the ordinary differential equation

$$\frac{1}{Pr}\frac{d^4\hat{\theta}}{d\xi^4} - s\left(1 + \frac{1}{Pr}\right)\frac{d^2\hat{\theta}}{d\xi^2} + (s^2+1)\hat{\theta} + \lim_{\tau\to 0}\left[\left(1+\frac{1}{Pr}\right)\frac{\partial^2\theta}{\partial\xi^2} - \frac{\partial\theta}{\partial\tau} - s\theta\right] = 0, \tag{13}$$

where $\hat{\theta} \equiv \int_0^\infty \theta e^{-s\tau}\, d\tau$ is the Laplace transform of θ. The terms in square brackets vanish as $\tau \to 0$, and (13) reduces to

$$\frac{1}{Pr}\frac{d^4\hat{\theta}}{d\xi^4} - s\left(1 + \frac{1}{Pr}\right)\frac{d^2\hat{\theta}}{d\xi^2} + (s^2+1)\hat{\theta} = 0. \tag{14}$$

Restricting attention to the case where $Pr = 1$, (14) becomes

$$\frac{d^4\hat{\theta}}{d\xi^4} - 2s\frac{d^2\hat{\theta}}{d\xi^2} + (s^2+1)\hat{\theta} = 0. \tag{15}$$

Solutions of (15) are linear combinations of exponential forms $\exp(m\xi)$ with $m^4 - 2sm^2 + s^2 + 1 = 0$, or $m = \pm\sqrt{s \pm \mathrm{i}}$. In our problem, θ (and hence $\hat{\theta}$) must vanish as $\xi \to \infty$, so we reject the contributions from the two roots $+\sqrt{s \pm \mathrm{i}}$, and obtain

$$\hat{\theta} = a \exp(-\xi \sqrt{s + \mathrm{i}}) + b \exp(-\xi \sqrt{s - \mathrm{i}}). \tag{16}$$

Boundary conditions at the plate are used to evaluate the coefficients a and b. Condition (11a) transforms as $\hat{\theta}(0) = 1/s$, while $W(0, \tau) = 0$ applied in (9) yields $\partial\theta/\partial\tau(0, \tau) = \partial^2\theta/\partial\xi^2(0, \tau)$, which transforms as $s\hat{\theta}(0) = \mathrm{d}^2\hat{\theta}/\mathrm{d}\xi^2(0)$. Applying these conditions in (16) yields $a + b = 1/s$ and $\mathrm{i}(a - b) = 0$, which are solved as $a = b = 1/(2s)$. The expression for θ then follows from the inverse transform of (16) as:

$$\theta = \tfrac{1}{2} L^{-1}\left[\frac{1}{s}\exp(-\xi\sqrt{s + \mathrm{i}})\right] + \tfrac{1}{2} L^{-1}\left[\frac{1}{s}\exp(-\xi\sqrt{s - \mathrm{i}})\right], \tag{17}$$

where L^{-1} is the inverse Laplace transform operator. The integration theorem, $L^{-1}[g(s)/s] = \int_0^\tau G(\tau')\,\mathrm{d}\tau'$, [where $G(\tau) = L^{-1}g(s)$], shifting theorem, $L^{-1}g(s \pm \mathrm{i}) = \exp(\mp \mathrm{i}\tau)G(\tau)$, and tabulated result $g(s) = \exp(-\xi\sqrt{s}) \leftrightarrow G(\tau) = \frac{\xi}{2\sqrt{\pi}\tau^{3/2}}\exp(-\xi^2/4\tau)$ (Doetsch 1961), lead to

$$L^{-1}\left[\frac{1}{s}\exp(-\xi\sqrt{s \pm \mathrm{i}})\right] = \frac{\xi}{2\sqrt{\pi}}\int_0^\tau \frac{1}{\tau'^{3/2}}\exp\left(\mp \mathrm{i}\tau' - \frac{\xi^2}{4\tau'}\right)\mathrm{d}\tau', \tag{18}$$

and so (17) becomes

$$\theta(\xi, \tau) = \frac{\xi}{2\sqrt{\pi}}\int_0^\tau \frac{\cos\tau'}{\tau'^{3/2}}\exp\left(-\frac{\xi^2}{4\tau'}\right)\mathrm{d}\tau'. \tag{19}$$

When evaluating (19) at $\xi = 0$, the integral should be carefully evaluated as $\xi \to 0$ rather than just applying $\xi = 0$ (see comments in Doetsch 1961, p. 169). To see more readily that the boundary condition $\theta(0, \tau) = 1$ is satisfied by (19), change the integration variable to $\lambda \equiv \xi/(2\sqrt{\tau'})$ to obtain

$$\theta(\xi, \tau) = \frac{2}{\sqrt{\pi}}\int_{\xi/(2\sqrt{\tau})}^\infty \cos\left(\frac{\xi^2}{4\lambda^2}\right)\exp(-\lambda^2)\,\mathrm{d}\lambda. \tag{20}$$

Letting $\xi \to 0$ in (20) yields the desired result,

$$\lim_{\xi \to 0}\theta(\xi, t) = \frac{2}{\sqrt{\pi}}\int_0^\infty \exp(-\lambda^2)\,\mathrm{d}\lambda = \mathrm{erf}(\infty) = 1.$$

Of special interest is the plate heat flux, obtained from (20) as

$$-\frac{\partial\theta}{\partial\xi}(0, \tau) = \frac{\cos\tau}{\sqrt{\pi\tau}} + \lim_{\xi \to 0}\int_{\xi/(2\sqrt{\tau})}^\infty \frac{\xi}{\sqrt{\pi}\lambda^2}\sin\left(\frac{\xi^2}{4\lambda^2}\right)\exp(-\lambda^2)\,\mathrm{d}\lambda. \tag{21}$$

Changing the integration variable in (21) to $\chi \equiv \xi/(\sqrt{2\pi}\lambda)$, we see that as $\xi \to 0$ the integral approaches $\sqrt{2}S(\infty)$, where $S(\infty) \equiv \int_0^\infty \sin(\pi\chi^2/2)\,\mathrm{d}\chi$ is a Fresnel sine integral (Abramowitz & Stegun 1964). Since $S(\infty) = 1/2$, (21) becomes

$$-\frac{\mathrm{d}\theta}{\mathrm{d}\xi}(0, \tau) = \frac{\cos\tau}{\sqrt{\pi\tau}} + \frac{1}{\sqrt{2}}. \tag{22}$$

Thus, the plate heat flux is infinite at $\tau = 0$, and undergoes a decaying oscillation as it approaches $1/\sqrt{2}$ in the limit $\tau \to \infty$.

The vertical velocity field can be calculated as a residual from (9) with θ supplied from (19). After integration by parts and suitable rearrangement, we obtain

$$W(\xi,\tau)=\frac{\xi}{2\sqrt{\pi}}\int_0^{\tau}\frac{\sin\tau'}{\tau'^{3/2}}\exp\left(-\frac{\xi^2}{4\tau'}\right)\mathrm{d}\tau'. \tag{23}$$

A change of integration variable to $\lambda\equiv\xi/(2\sqrt{\tau'})$ yields the alternative form:

$$W(\xi,\tau)=\frac{2}{\sqrt{\pi}}\int_{\xi/(2\sqrt{\tau})}^{\infty}\sin\left(\frac{\xi^2}{4\lambda^2}\right)\exp(-\lambda^2)\,\mathrm{d}\lambda. \tag{24}$$

The locations of peak acceleration $\xi_{max\dot{W}}$ and peak rate of temperature change $\xi_{max\dot{\theta}}$ defined by $(\partial^2 W/\partial\xi\,\partial\tau)(\xi_{max\dot{W}},\tau)=0$ and $(\partial^2\theta/\partial\xi\,\partial\tau)(\xi_{max\dot{\theta}},\tau)=0$, respectively, are found to coincide with each other, $\xi_{max\dot{W}}=\xi_{max\dot{\theta}}=\sqrt{2\tau}$. These locations propagate along the ξ-axis with a speed $1/\sqrt{2\tau}$ that decreases monotonically with time but is infinite at $\tau=0$. The corresponding peak rates are

$$\frac{\partial W}{\partial\tau}(\xi_{max\dot{W}},\tau)=\frac{\exp\left(-\frac{1}{2}\right)}{\sqrt{2\pi}}\frac{\sin\tau}{\tau}\simeq 0.24197\frac{\sin\tau}{\tau}, \tag{25}$$

$$\frac{\partial\theta}{\partial\tau}(\xi_{max\dot{\theta}},\tau)=\frac{\exp\left(-\frac{1}{2}\right)}{\sqrt{2\pi}}\frac{\cos\tau}{\tau}\simeq 0.24197\frac{\cos\tau}{\tau}. \tag{26}$$

Thus, the peak acceleration and peak rate of temperature change oscillate with an overall temporal decay (values approaching 0 as $\tau\to\infty$). The peak rate of temperature change leads the peak acceleration by a quarter period ($\cos\tau$ versus $\sin\tau$). Since the non-dimensional period 2π corresponds to a dimensional period of $2\pi(T_r/(g\gamma))^{1/2}$, stronger stratifications (larger γ) are associated with higher frequencies. Since the maximum value of $\sin\tau/\tau$ is 1 (at $\tau=0$), the peak acceleration never exceeds its initial value of 0.24197. In contrast, the peak rate of temperature change is infinite at $\tau=0$.

Contour plots of θ and W as a function of ξ and τ are presented in figure 1. These plots show the boundary-layer character of the solutions and the oscillatory approach to steady-state conditions. The plots were constructed by numerically evaluating the integrals in (19) and (23) with the trapezoidal formula with time step $\Delta\tau=0.001$. Such a high temporal resolution was required for an accurate evaluation of θ near $\xi=0$ because of the singular nature of the integral in (19) for $\xi\to 0$. The value $\theta=1$, which the solution (19) approaches as $\xi\to 0$, was explicitly imposed at $\xi=0$.

The related transient flow of a stratified fluid in a vertical channel induced by a step-change in the temperature of the two sidewalls (Park & Hyun 1998; Park 2001) also exhibited an oscillatory approach to the steady state for $Pr=1$. For $Pr\neq 1$, the approach to the steady state was oscillatory for Rayleigh numbers exceeding $(Pr-1)^2\pi^4/(4Pr)$, but was non-oscillatory for smaller Rayleigh numbers. The Prandtl number sensitivity of our single-plate convective flows will be examined in a later investigation.

3.2. *Steady-state solution*

The steady-state perturbation temperature $\theta_s(\xi)$ satisfies (12) with time derivatives neglected. Solutions are linear combinations of the form $\theta_s=\exp(m\xi)$ provided $m=\pm(1\pm\mathrm{i})Pr^{1/4}/\sqrt{2}$. Rejecting the solutions associated with roots with positive real parts, we obtain

$$\theta_s(\xi)=c\exp\left[-\xi(1-\mathrm{i})Pr^{1/4}/\sqrt{2}\right]+\mathrm{d}\exp\left[-\xi(1+\mathrm{i})Pr^{1/4}/\sqrt{2}\right]. \tag{27}$$

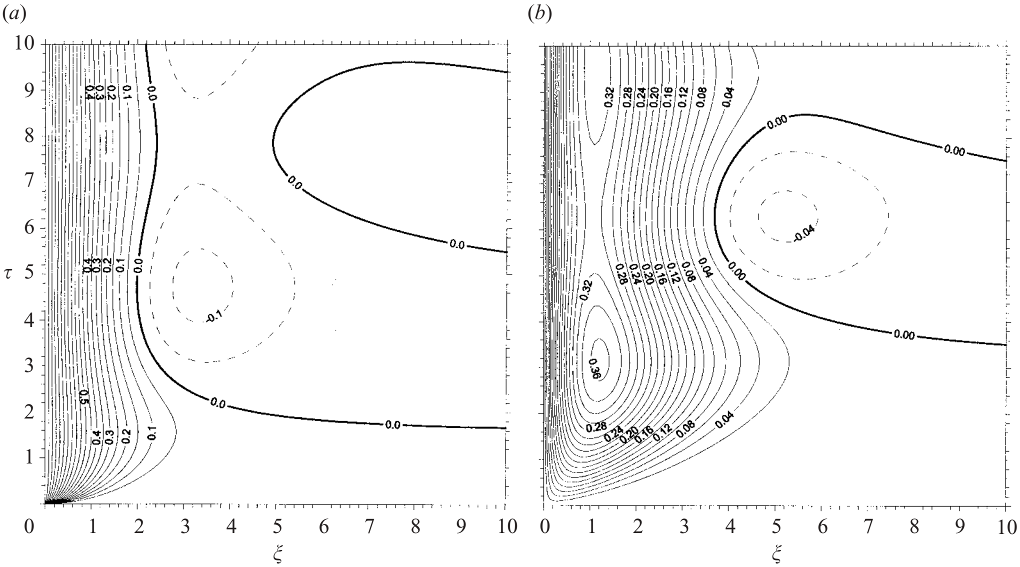

FIGURE 1. Contours of (*a*) $\theta(\xi, \tau)$ from (19), and (*b*) $W(\xi, \tau)$ from (23) for the case of an impulsive (step) change in plate perturbation temperature. The contour increment is 0.02 in $W(\xi, \tau)$ and 0.05 in $\theta(\xi, \tau)$. Negative contours are dashed.

The no-slip condition applied in the steady-state version of (9) yields $\mathrm{d}^2\theta_s/\mathrm{d}\xi^2(0) = 0$. Applying this condition and (11*a*) in (27), we find that $c = d = 1/2$, and so

$$\theta_s(\xi) = \cos\left(\xi Pr^{1/4}/\sqrt{2}\right)\exp\left(-\xi Pr^{1/4}/\sqrt{2}\right). \tag{28}$$

The steady-state vertical velocity is readily found to be

$$W_s(\xi) = \frac{1}{\sqrt{Pr}}\sin\left(\xi Pr^{1/4}/\sqrt{2}\right)\exp\left(-\xi Pr^{1/4}/\sqrt{2}\right). \tag{29}$$

Equations (28) and (29) were obtained by Gill (1966), who recognized their correspondence to the Prandtl (1952) solution for one-dimensional mountain and valley winds along a sloping planar boundary in a stratified atmosphere. Because of its boundary-layer structure, this solution is sometimes refered to as a buoyancy layer (e.g. Bergholz 1978). With a suitable change of parameters this same solution describes the along-slope flow and salinity (density) perturbations in an oceanic mixing layer at a sloping sidewall (Phillips 1970; Wunsch 1970). In the oceanic context, the flow is generated solenoidally by isopycnals that are forced to approach the sloping boundary at a right angle (zero normal flux condition). Because of the mathematical analogy between stratified and rotating flows (Veronis 1970), this same solution (with suitable changes of variables and parameters) also describes the familiar Ekman flow of a homogeneous viscous rotating fluid in the presence of an imposed wind stress or a stationary horizontal boundary (Holton 1992; Kundu & Cohen 2002).

The peak vertical velocity in the steady state occurs at a non-dimensional distance $\delta = \pi\sqrt{2}Pr^{-1/4}/4 \simeq 1.1107 Pr^{-1/4}$ from the plate, and has the value $W_s(\delta) = (1/\sqrt{2Pr})\exp(-\pi/4) \simeq 0.32239/\sqrt{Pr}$. The perturbation temperature drops to approximately one third of its plate value at this location. Thus, δ is a convenient measure of the boundary-layer thickness. For the particular case of an isothermal environment ($\mathrm{d}T_\infty/\mathrm{d}z = 0$, so $\gamma = g/c_p$) with $T_0' = 2\,\mathrm{K}$, $Pr = 1$, $T_r = 293\,\mathrm{K}$, $g = 9.8\,\mathrm{m\,s^{-2}}$,

$c_p = 1004\,\mathrm{J\,kg^{-1}\,K^{-1}}$, and $\nu = 1.5\times10^{-5}\,\mathrm{m^2\,s^{-1}}$, the dimensional boundary-layer thickness $\pi\sqrt{2\nu}(g\gamma Pr/T_r)^{-1/4}/4$ is approximately 0.03 m, and the peak vertical velocity is approximately $1.2\,\mathrm{m\,s^{-1}}$. If we consider a large ambient stratification, $\mathrm{d}T_\infty/\mathrm{d}z = 1\,\mathrm{K\ m^{-1}}$, with all other parameters unchanged, the dimensional boundary-layer thickness is reduced to 0.01 m while the peak vertical velocity is reduced to $0.12\,\mathrm{m\,s^{-1}}$.

In the Appendix, we show that as $\tau\to\infty$ the unsteady solutions (19) and (23) approach the steady-state solutions (28) and (29) with $Pr = 1$.

4. Sudden application of a plate heat flux

4.1. *Solution by Laplace transforms*

Now consider the case where a plate heat flux is suddenly applied at $\tau = 0$. We again obtain (12) for θ and (16) for $\hat{\theta}$ (with $Pr = 1$). The heat flux condition (11*b*) transforms as $\mathrm{d}\hat{\theta}/\mathrm{d}\xi(0) = -1/s$, while the no-slip condition applied in (9) yields $\partial\theta/\partial\tau(0,\tau) = \partial^2\theta/\partial\xi^2(0,\tau)$, which transforms as $s\hat{\theta}(0) = \mathrm{d}^2\hat{\theta}/\mathrm{d}\xi^2(0)$. Applying these conditions in (16) yields two equations for a and b that are solved as $a = b = (\sqrt{s+\mathrm{i}} - \sqrt{s-\mathrm{i}})/(2\mathrm{i}s)$, and so

$$\hat{\theta} = \frac{1}{2\mathrm{i}}(\sqrt{s+\mathrm{i}} - \sqrt{s-\mathrm{i}})\left[\frac{1}{s}\exp(-\xi\sqrt{s+\mathrm{i}}) + \frac{1}{s}\exp(-\xi\sqrt{s-\mathrm{i}})\right]. \tag{30}$$

The inverse transform of (30) is evaluated with the convolution theorem used in conjunction with (18), that is, the inverse transform of terms enclosed by square parentheses in (30), and the tabulated result:

$$f(s) = \sqrt{s+\mathrm{i}} - \sqrt{s-\mathrm{i}} \leftrightarrow F(\tau) = \frac{1}{2\sqrt{\pi}\tau^{3/2}}[\exp(\mathrm{i}\tau) - \exp(-\mathrm{i}\tau)] = \frac{\mathrm{i}\sin\tau}{\sqrt{\pi}\tau^{3/2}}.$$

We obtain

$$\theta(\xi,\tau) = \frac{\xi}{2\pi}\int_0^\tau \frac{\sin(\tau-\tau')}{(\tau-\tau')^{3/2}}\int_0^{\tau'}\frac{\cos\tau''}{\tau''^{3/2}}\exp\left(-\frac{\xi^2}{4\tau''}\right)\mathrm{d}\tau''\,\mathrm{d}\tau', \tag{31}$$

where τ' and τ'' are dummy variables.

Comparing the solution (31) for the sudden application of a plate heat flux with the solution (19) for the step-change in plate temperature (written with $\theta_{\Delta temp}$ in place of θ), we have

$$\theta(\xi,\tau) = \frac{1}{\sqrt{\pi}}\int_0^\tau \frac{\sin(\tau-\tau')}{(\tau-\tau')^{3/2}}\theta_{\Delta temp}(\xi,\tau')\,\mathrm{d}\tau', \tag{32}$$

which means that at any location ξ, the solution associated with a sudden application of a plate heat flux is a weighted average over time of the solution associated with a step-change in plate temperature.

To obtain the plate temperature perturbation in the heat flux problem, consider the limit of (31) as $\xi\to0$, or, more simply, consider $\xi = 0$ in (32):

$$\theta(0,\tau) = \frac{1}{\sqrt{\pi}}\int_0^\tau \frac{\sin(\tau-\tau')}{(\tau-\tau')^{3/2}}\,\mathrm{d}\tau'.$$

Changing variables to $\chi \equiv \sqrt{2(\tau-\tau')/\pi}$ and integrating by parts, we obtain

$$\theta(0,\tau) = -\frac{2\sin\tau}{\sqrt{\pi\tau}} + 2^{3/2}C(\sqrt{2\tau/\pi}), \tag{33}$$

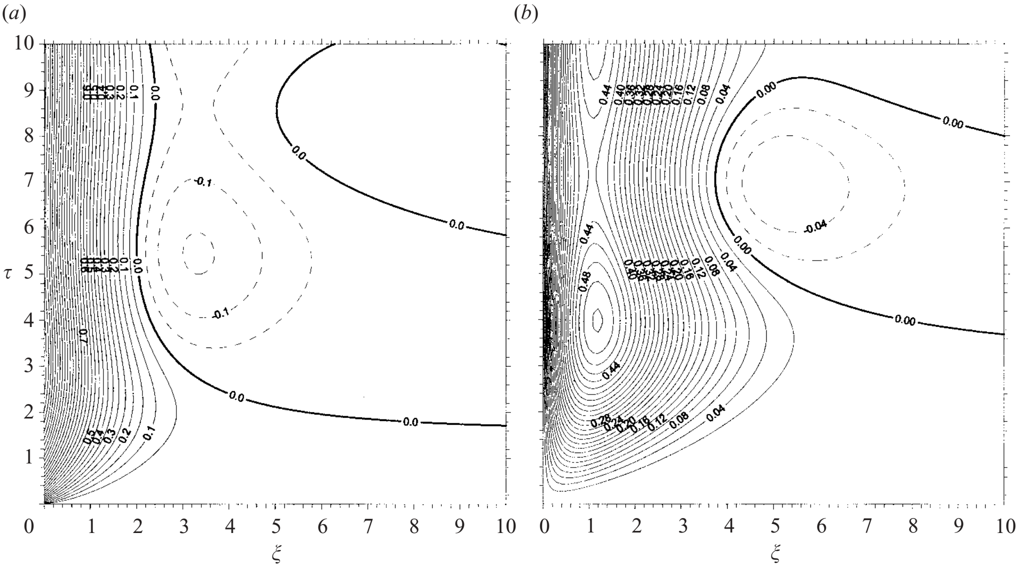

FIGURE 2. Contours of (*a*) $\theta(\xi,\tau)$ from (31), and (*b*) $W(\xi,\tau)$ from (34) for the case of a suddenly applied plate heat flux. Contour increments are as in figure 1.

where $C(\chi) \equiv \int_0^{\chi} \cos(\pi\chi'^2/2)\,\mathrm{d}\chi'$ is a Fresnel cosine integral (Abramowitz & Stegun 1964). Since $C(\infty) = 1/2$, the plate temperature perturbation approaches $\sqrt{2}$ as $\tau \to \infty$.

The vertical velocity is again obtained as a residual from (9):

$$W(\xi,\tau) = \frac{\xi}{2\pi} \int_0^{\tau} \frac{\sin(\tau-\tau')}{(\tau-\tau')^{3/2}} \int_0^{\tau'} \frac{\sin\tau''}{\tau''^{3/2}} \exp\left(-\frac{\xi^2}{4\tau''}\right) \mathrm{d}\tau''\mathrm{d}\tau'. \tag{34}$$

Comparing (34) with (23), we see that W can be expressed as a weighted average of the solution associated with a step-change in plate perturbation temperature (denoted by $W_{\Delta temp}$),

$$W(\xi,\tau) = \frac{1}{\sqrt{\pi}} \int_0^{\tau} \frac{\sin(\tau-\tau')}{(\tau-\tau')^{3/2}} W_{\Delta temp}(\xi,\tau')\,\mathrm{d}\tau'. \tag{35}$$

Contour plots of θ and W obtained by numerical evaluation of (31) and (34) are presented in figure 2. Equation (32) was used to evaluate θ at $\xi = 0$. As in the previous problem, a high temporal resolution ($\Delta\tau = 0.001$) is required near the plate because of the singular nature of the integral in the solution for θ. Qualitatively, we see that the solutions for W and θ in this heat flux case are similar to the solutions obtained for the plate temperature perturbation problem (figure 1).

4.2. *Steady state*

As in the previous example, the perturbation temperature in the steady state satisfies (27). The heat flux condition (11*b*) and no-slip condition applied in the steady-state version of (9) yield $c = d = 1/(\sqrt{2}Pr^{1/4})$, and so

$$\theta_s(\xi) = \frac{\sqrt{2}}{Pr^{1/4}} \cos\left(\xi Pr^{1/4}/\sqrt{2}\right) \exp\left(-\xi Pr^{1/4}/\sqrt{2}\right). \tag{36}$$

Similarly, we find the solution of the steady-state vertical velocity is

$$W_s(\xi) = \frac{\sqrt{2}}{Pr^{3/4}} \sin\left(\xi Pr^{1/4}/\sqrt{2}\right) \exp\left(-\xi Pr^{1/4}/\sqrt{2}\right). \tag{37}$$

These expressions are identical to the solution (28) and (29) associated with a step-change in plate temperature perturbation, apart from a multiplicatative factor of $\sqrt{2}/Pr^{1/4}$. In particular, the peak vertical velocity still occurs at a distance $\delta = \pi\sqrt{2}Pr^{-1/4}/4$ from the plate, but now has increased to the value

$$W_s(\delta) = \frac{1}{Pr^{3/4}} \exp(-\pi/4) \simeq 0.45593\ Pr^{-3/4}.$$

In the Appendix, we show that, as $\tau \to \infty$, the unsteady solutions (31) and (34) approach the steady-state solutions (36) and (37) with $Pr = 1$.

5. Comparison with the classical solutions

The solutions derived in §§3 and 4 are now compared with the classical solutions in which pressure work is neglected and the environment is considered to be isothermal (e.g. Goldstein & Briggs 1964). To facilitate the comparison, we non-dimensionalize the classical solutions in the same manner as our new solutions. The classical solutions are expressed in terms of the complementary error function

$$\mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-\lambda^2)\,\mathrm{d}\lambda$$

and its integrals,

$$\mathrm{i\,erfc}(x) \equiv \int_x^\infty \mathrm{erfc}(\lambda)\,\mathrm{d}\lambda, \quad \mathrm{i^2erfc}(x) \equiv \int_x^\infty \mathrm{i\ erfc}(\lambda)\,\mathrm{d}\lambda.$$

To aid in their numerical evaluation, the solutions are also rewritten using the recurrence relation 7.2.5 of Abramowitz & Stegun (1964), and with the erfc integral rewritten with time as the integration variable.

The classical solution non-dimensionalized with (6) and (7*a*) for the case of a step-change in plate temperature and $Pr = 1$ is,

$$\theta = \mathrm{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right) = \frac{\xi}{2\sqrt{\pi}} \int_0^\tau \frac{1}{\tau'^{3/2}} \exp\left(-\frac{\xi^2}{4\tau'}\right) \mathrm{d}\tau', \tag{38}$$

$$W = \xi\sqrt{\tau}\,\mathrm{i\,erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right) = -\frac{\xi^3}{4\sqrt{\pi}} \int_0^\tau \frac{1}{\tau'^{3/2}} \exp\left(-\frac{\xi^2}{4\tau'}\right) \mathrm{d}\tau' + \xi\frac{\sqrt{\tau}}{\sqrt{\pi}} \exp\left(-\frac{\xi^2}{4\tau}\right). \tag{39}$$

Contour plots of these classical solutions are presented in figure 3 (compare with new solutions in figure 1, but note that contour increment for W is different in the two figures). The behaviour of the new and classical solutions at a fixed distance from the plate, $\xi = 1$, is presented in figure 5. For small times, potential temperature stratification has little impact on the flow; the new solution and the classical solution are in close agreement for $\tau < 1$. However, with larger τ, stratification becomes important and the solutions rapidly diverge. In the classical model, there is an inexorable conductively driven spread of the thermal disturbance outward throughout

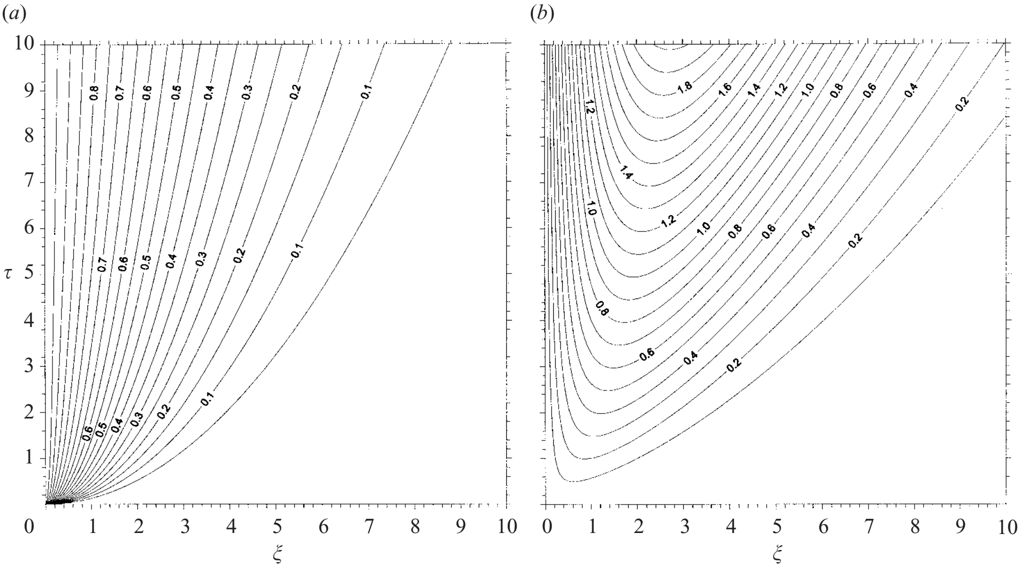

FIGURE 3. Contours of classical solution for (*a*) $\theta(\xi, \tau)$ from (38), and (*b*) $W(\xi, \tau)$ from (39) for the case of an impulsive (step) change in plate perturbation temperature in a fluid with no temperature stratification and with pressure work neglected. The contour increment is 0.1 in $W(\xi, \tau)$ and 0.05 in $\theta(\xi, \tau)$.

the domain, and the associated buoyancy force drives a perpetual fluid acceleration. In contrast, the negative feedback associated with potential temperature advection in a stably stratified environment inhibits the spread of the disturbance in the new model, and the solution approaches a steady state.

The classical solution non-dimensionalized with (6) and (7*b*) for the case of a suddenly imposed plate heat flux and $Pr = 1$ is

$$\theta = 2\sqrt{\tau}\,\mathrm{i\,erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right) = -\frac{\xi^2}{2\sqrt{\pi}}\int_0^{\tau}\frac{1}{\tau'^{3/2}}\exp\left(-\frac{\xi^2}{4\tau'}\right)\mathrm{d}\tau' + \frac{2\sqrt{\tau}}{\sqrt{\pi}}\exp\left(-\frac{\xi^2}{4\tau}\right), \tag{40}$$

$$\begin{aligned} W &= 2\xi\tau\,\mathrm{i}^2\,\mathrm{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right) \\ &= \frac{\xi^2}{4\sqrt{\pi}}\left(\frac{\xi^2}{2}+\tau\right)\int_0^{\tau}\frac{1}{\tau'^{3/2}}\exp\left(-\frac{\xi^2}{4\tau'}\right)\mathrm{d}\tau' - \frac{\xi^2\sqrt{\tau}}{2\sqrt{\pi}}\exp\left(-\frac{\xi^2}{4\tau}\right). \end{aligned} \tag{41}$$

Contour plots of these classical solutions are presented in figure 4 (compare with new solutions in figure 2, again noting the difference in contour increment for W) and the temporal behaviour of the solutions at the fixed distance $\xi = 1$ is presented in figure 5. Again, there is significant divergence between the classical and new solutions for $\tau > 1$.

Figure 6 depicts the evolution of the temperature perturbation at the plate in the case of the suddenly imposed plate heat flux for the new solution (32) and the classical solution (40). This figure clearly shows the close agreement for $\tau < 1$, and the differing behaviour at larger times.

A cross-section of the perturbation temperature and vertical velocity at $\tau = 2$ is presented in figure 7. In the cases of both the impulsively changed plate temperature

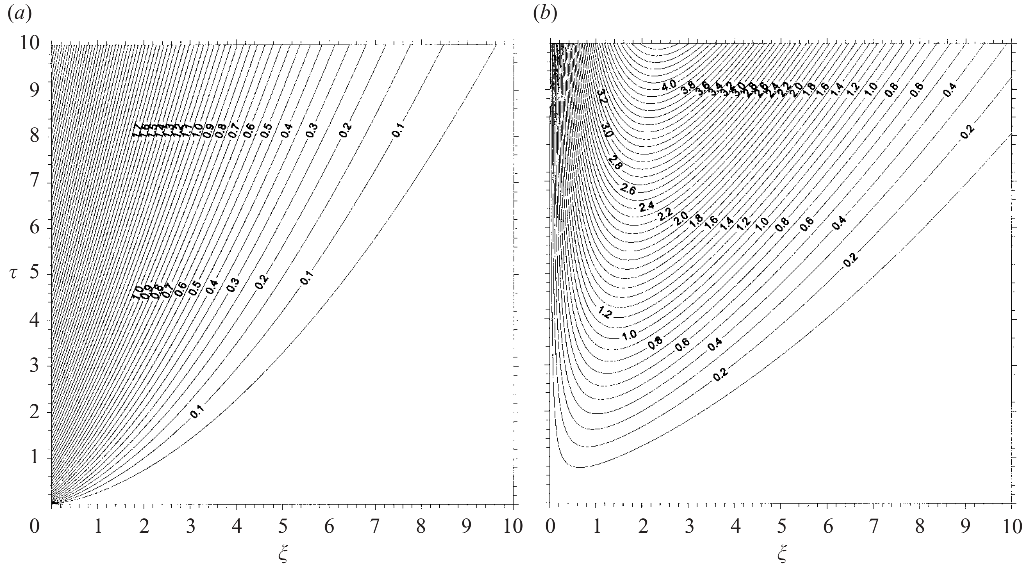

FIGURE 4. Contours of classical solution for (a) $\theta(\xi, \tau)$ from (40), and (b) $W(\xi, \tau)$ from (41) for the case of a suddenly applied plate heat flux in a fluid with no temperature stratification and with pressure work neglected. Contour increments are as in figure 3.

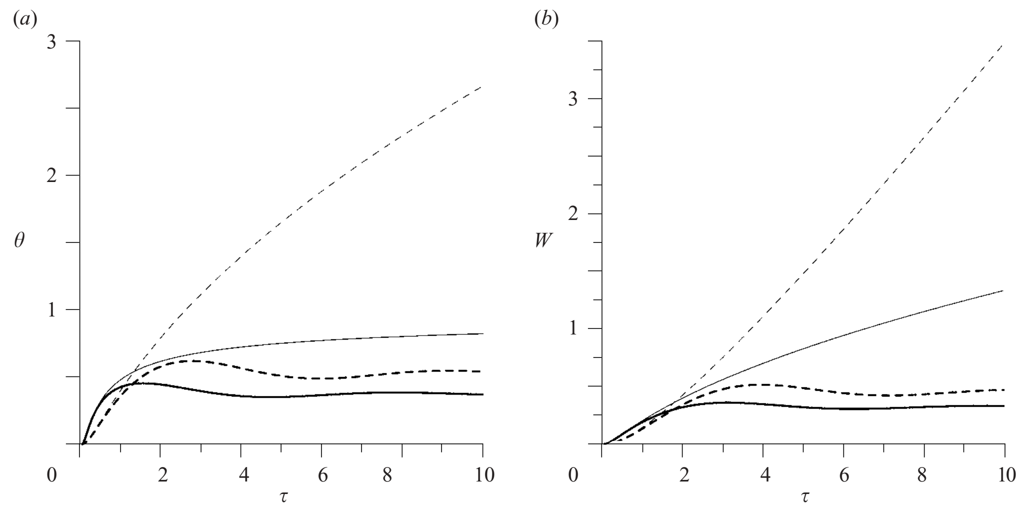

FIGURE 5. Temporal variations of (a) $\theta(\xi, \tau)$ and (b) $W(\xi, \tau)$ at the dimensionless distance from the plate $\xi = 1$. The solid line presents the classical solution and heavy solid line presents the new solution for the case of an impulsive (step) change in plate perturbation temperature. The dashed line presents the classical solution and the heavy dashed line presents the new solution for the case of suddenly applied plate heat flux.

perturbation and suddenly imposed plate heat flux, the classical model predicts larger temperature perturbations and larger vertical velocities than in the corresponding stratified model.

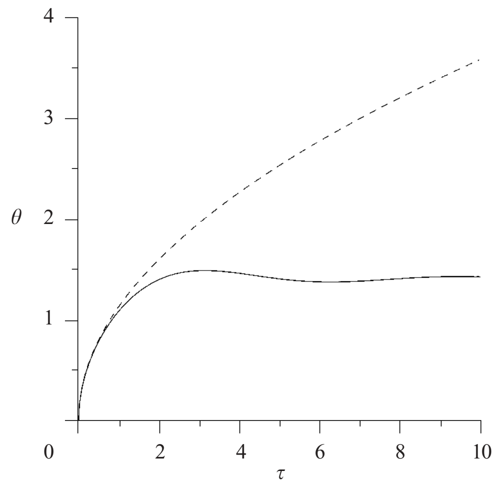

FIGURE 6. Perturbation temperature at the plate, $\theta(0, \tau)$, for the case of a suddenly applied plate heat flux. The solid line presents the new solution and the dashed line presents the classical solution.

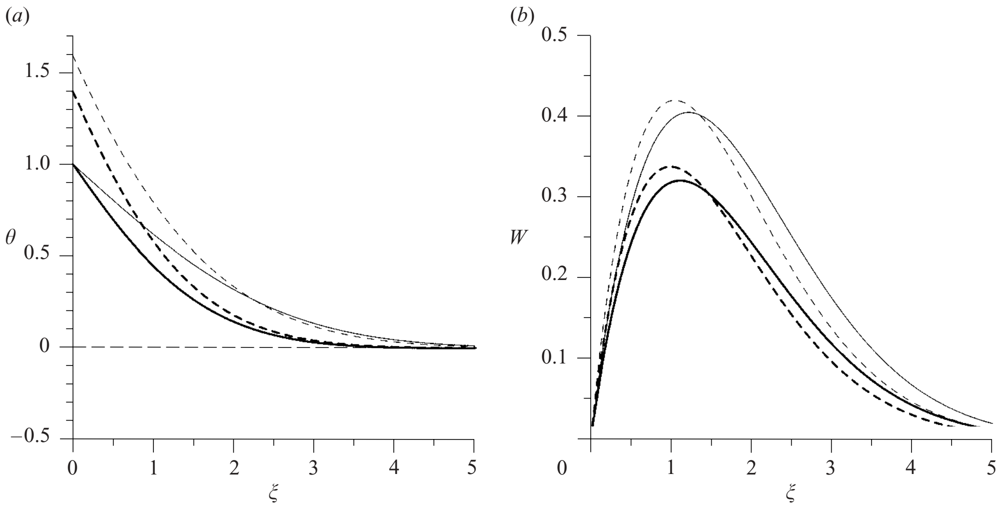

FIGURE 7. Cross-sections of (*a*) $\theta(\xi, \tau)$ and (*b*) $W(\xi, \tau)$ at dimensionless time $\tau = 2$. The solid line presents the classical solution and the heavy solid line presents the new solution for the case of an impulsive (step) change in plate perturbation temperature. The dashed line presents the classical solution and the heavy dashed line presents the new solution for the case of suddenly applied plate heat flux.

6. Convection driven by arbitrary temporal variations of plate thermal properties

It is straightforward to obtain solutions for one-dimensional plate convection driven by arbitrary temporal variations of plate perturbation temperature or plate heat flux. As in the previous sections, the fluid is at rest with zero horizontal temperature gradient until the onset of the thermal disturbance at $t = 0$. Again, we restrict attention to a Prandtl number of unity.

First, consider the case of a plate perturbation temperature that varies in an arbitrary manner, $T'(0,t)=f(t)$. Letting T_0' denote the maximum value of $f(t)$ on the interval $t\in(0,\infty)$, and introducing the non-dimensional perturbation temperature function $F(t)\equiv f(t)/T_0'$, we can write the plate perturbation temperature as $T'(0,t)=T_0'F(t)$, where $F(t)$ attains a maximum value of 1 on the interval $t\in(0,\infty)$ but is otherwise arbitrary. The non-dimensionalization (6) and (7*a*) (with T_0' now interpreted as the maximum plate perturbation temperature) again yields (8)–(10) which lead to (15) and (16). The plate perturbation temperature condition $\theta(0,\tau)=F(\tau)$ transforms as $\hat{\theta}(0)=\hat{F}(s)$, where $\hat{F}(s)$ is the Laplace transform of $F(\tau)$, while the no-slip condition applied in (9) again leads to $s\hat{\theta}(0)=\mathrm{d}^2\hat{\theta}/\mathrm{d}\xi^2(0)$. Applying these conditions in (16) yields $a=b=\hat{F}(s)/2$, and so

$$\hat{\theta}=\tfrac{1}{2}\hat{F}(s)[\exp(-\xi\sqrt{s+\mathrm{i}}+\exp(-\xi\sqrt{s-\mathrm{i}})]. \tag{42}$$

Application of the convolution theorem then yields

$$\theta(\xi,\tau)=\frac{\xi}{2\sqrt{\pi}}\int_0^\tau \frac{F(\tau-\tau')\cos\tau'}{\tau'^{3/2}}\exp\left(-\frac{\xi^2}{4\tau'}\right)\mathrm{d}\tau'. \tag{43}$$

Equation (43) can be rewritten in terms of the solution (19) (now denoted by $\theta_{\Delta temp}$) for the temperature perturbation induced by a step-change in plate temperature perturbation,

$$\theta(\xi,\tau)=\int_0^\tau F(\tau-\tau')\frac{\partial\theta_{\Delta temp}}{\partial\tau'}(\xi,\tau')\,\mathrm{d}\tau'. \tag{44}$$

The form of (44) is reminiscent of the superposition (Duhamel) solutions for classical boundary-value problems of heat conduction (Carslaw & Jaeger 1959; Beck *et al.* 1992).

In a similar manner, the vertical velocity is obtained as

$$W(\xi,\tau)=\frac{\xi}{2\sqrt{\pi}}\int_0^\tau \frac{F(\tau-\tau')\sin\tau'}{\tau'^{3/2}}\exp\left(-\frac{\xi^2}{4\tau'}\right)\mathrm{d}\tau' \tag{45}$$

or

$$W(\xi,\tau)=\int_0^\tau F(\tau-\tau')\frac{\partial W_{\Delta temp}}{\partial\tau'}(\xi,\tau')\,\mathrm{d}\tau', \tag{46}$$

where $W_{\Delta temp}$ is given by (23).

Now consider the flow induced by a plate heat flux of the form $-k\partial T'/\partial x(0,t)=QF(t)$, where the non-dimensional heat flux function $F(t)$ attains a maximum value of 1 on the interval $t\in(0,\infty)$, but is otherwise arbitrary, and Q is the maximum plate heat flux on the interval $t\in(0,\infty)$. The non-dimensionalization (6) and (7*b*) again yields (8)–(10) which lead to (15) and (16). The non-dimensional heat flux $\partial\theta/\partial\xi(0,\tau)=-F(\tau)$ transforms as $\mathrm{d}\hat{\theta}/\mathrm{d}\xi(0)=-\hat{F}(s)$ where $\hat{F}(s)$ is the Laplace transform of $F(\tau)$. Application of this condition and the transformed no-slip condition in (16) lead to $a=b=\hat{F}(s)(\sqrt{s+\mathrm{i}}-\sqrt{s-\mathrm{i}})/(2\mathrm{i})$, and so

$$\hat{\theta}=\frac{\hat{F}(s)}{2\mathrm{i}}(\sqrt{s+\mathrm{i}}-\sqrt{s-\mathrm{i}})[\exp(-\xi\sqrt{s+\mathrm{i}})+\exp(-\xi\sqrt{s-\mathrm{i}})]. \tag{47}$$

The solution for θ then follows from the convolution theorem and results from § 4 as

$$\theta(\xi,\tau)=\frac{\xi}{2\pi}\int_0^\tau F(\tau-\tau')\int_0^{\tau'}\frac{\sin(\tau'-\tau'')}{(\tau'-\tau'')^{3/2}}\frac{\cos\tau''}{\tau''^{3/2}}\exp\left(-\frac{\xi^2}{4\tau''}\right)\mathrm{d}\tau''\,\mathrm{d}\tau'. \tag{48}$$

In view of the solution (31) for the perturbation temperature associated with sudden application of a plate heat flux (now denoted by $\theta_{\Delta flux}$), (48) can be put in the form of the superposition integral

$$\theta(\xi,\tau)=\int_0^{\tau}F(\tau-\tau')\frac{\partial\theta_{\Delta flux}}{\partial\tau'}(\xi,\tau')\,\mathrm{d}\tau'. \tag{49}$$

In a similar manner, the vertical velocity is obtained as

$$W(\xi,\tau)=\frac{\xi}{2\pi}\int_0^{\tau}F(\tau-\tau')\int_0^{\tau'}\frac{\sin(\tau'-\tau'')}{(\tau'-\tau'')^{3/2}}\frac{\sin\tau''}{\tau''^{3/2}}\exp\left(-\frac{\xi^2}{4\tau''}\right)\mathrm{d}\tau''\mathrm{d}\tau' \tag{50}$$

or

$$W(\xi,\tau)=\int_0^{\tau}F(\tau-\tau')\frac{\partial W_{\Delta flux}}{\partial\tau'}(\xi,\tau')\,\mathrm{d}\tau', \tag{51}$$

where $W_{\Delta flux}$ is defined by (34).

7. Summary and discussion

This study revisits one of the simplest scenarios of natural convection, the one-dimensional (parallel) convectively driven flow of a viscous fluid along an infinite vertical plate. Our model refines the classical theory by including the pressure work term in the thermodynamic energy equation and extends the theory by making provision for vertical temperature advection. The two terms are of the same form and can be conveniently combined into a single term for advection of the potential temperature. With attention restricted to a Prandtl number of unity, exact solutions of the viscous equations of motion are obtained for flows driven by an impulsively changed plate perturbation temperature, a suddenly imposed plate heat flux, and arbitrary temporal variations of plate perturbation temperature or plate heat flux. The considered thermodynamic processes introduce a negative feedback mechanism whereby warm fluid rises and cools relative to the environment, while cool fluid subsides and warms relative to the environment. More precisely, for the case of air parcels rising in an environment of positive temperature stratification $\mathrm{d}T_\infty/\mathrm{d}z>0$, the advection term accounts for progressively higher environmental temperatures encountered by the parcels (i.e. cooling of upward-displaced parcels relative to their environment), while the pressure work term accounts for expansional cooling. The negative feedback mechanism results in a flow that approaches a steady state at large times. In contrast, in the classical solutions where pressure work is neglected and there is no temperature stratification, the disturbance continues to spread outward from the plate and no steady state is approached. In these latter flows, the fluid experiences a persistent buoyancy-induced acceleration, and the vertical velocity grows without bound. It should be noted, however, that the pressure work term is generally quite small, and that the main interest in our solutions will probably be in the effects of temperature stratification. We have retained the pressure work term because it is convenient to do so and because the associated analytic solutions can potentially be used to validate numerical convection models that include that term.

We now briefly discuss the factors that bear on the applicability of our one-dimensional model. First, limitations imposed by the Boussinesq approximation will restrict the vertical extent that can legitimately be considered. Under typical atmospheric conditions, the Boussinesq approximation is a good approximation for flows with vertical length scales up to 1 km (Holton 1992). This imposes an upper bound on the vertical extent we should consider for our solution domain, at least for terrestrial applications. However, two-dimensional effects associated with the passage of the leading-edge signal will eventually terminate (locally) the one-dimensional regime. Moreover, in regions where the developing one-dimensional flow becomes unstable before passage of the leading-edge disturbance, the timing of the instability will control the duration of the one-dimensional regime. Although the stability problem for our transient solutions is beyond the scope of this study, the transient flow experiments of Joshi & Gebhart (1987), Brooker *et al.* (2000) and Patterson *et al.* (2002) suggest the likelihood of instability in cases of no ambient temperature stratification for all but the smallest thermal forcings, while the analyses of Gill & Davey (1969) and Bergholz (1978) for steady buoyancy layers in a stratified fluid suggest that stratification will exert a largely stabilizing influence on the flow.

Since differences between our new (stratified) solution and the classical solution only become apparent for $\tau > 1$, stratification effects will only become important if the flow is stable until at least $\tau = 1$, and at heights for which the leading-edge effect has not yet propagated by $\tau = 1$. For the parameters considered at the end of § 3, namely, 2 K plate temperature perturbation in an isothermal environment, $\tau = 1$ corresponds to a dimensional time of $t \approx 56$ s. Since the domain-maximum W increases roughly linearly from 0 at $\tau = 0$ to 0.2 at $\tau = 1$ (figure 1*b*), its average value over this period is approximately 0.1, and its corresponding dimensional average value is about $0.37\,\mathrm{m\,s^{-1}}$. The leading-edge disturbance is known to propagate at speeds generally higher than the fastest convective speeds in the boundary layer (Mahajan & Gebhart 1978; Joshi & Gebhart 1987; Daniels & Patterson 1997, 2001). Estimating the distance travelled by this disturbance with a speed 50% larger than the $0.37\,\mathrm{m\,s^{-1}}$ convective speed, the leading-edge disturbance would propagate about 30 m during this time period. Thus, for discrepancies between the classical and new solutions to become apparent, the flow would have to be stable for at least the first minute, and we should consider plates exceeding 30 m in height. On the other hand, if we consider the case of large ambient temperature stratification, $\mathrm{d}T_\infty/\mathrm{d}z = 1\,\mathrm{K\,m^{-1}}$, with all other parameters unchanged, $\tau = 1$ corresponds to a dimensional time of only $t \approx 5$ s. Over this time period, the dimensional average value of the domain-maximum vertical velocity is about $0.037\,\mathrm{m\,s^{-1}}$, and the leading-edge disturbance would propagate less than 0.3 m (again assuming a disturbance propagating 50% faster than the convective speed). Thus, for differences between the classical and new solutions to become apparent for this case of large ambient stratification, the flow would have to be stable for at least 5 s, and we should consider plates exceeding 0.3 m in height.

In summary, our analytic solutions provide a useful description of transient natural convection from an infinite vertical plate in a stratified fluid in regions where the leading-edge effect has not yet propagated and for times prior to the onset of any instabilities. These solutions can also be employed to validate numerical convection models. Although the stability of these flows is beyond the scope of the present investigation, these solutions can serve as a departure point (transient base state) for studies of waves and instabilities in heated vertical boundary layers in a stratified flow.

Appendix. Verification that the unsteady solutions approach the steady-state solutions as $\tau \to \infty$

Consider the solution for convection induced by an impulsively changed plate temperature. We wish to show that, as $\tau \to \infty$, the unsteady solutions (19) and (23) (valid for $Pr = 1$) approach the steady-state solutions (28) and (29) with $Pr = 1$.

Toward that end, rewrite (19) in the limit $\tau \to \infty$ as

$$\theta(\xi, \infty) = \mathrm{Re} \int_0^\infty \exp(\mathrm{i}\tau') \frac{\xi}{2\sqrt{\pi}\tau'^{3/2}} \exp\left(-\frac{\xi^2}{4\tau'}\right) \mathrm{d}\tau'. \tag{A 1}$$

This integral can be identified as the tabulated Laplace transform of $(\xi/2\sqrt{\pi}\tau'^{3/2}) \exp(-\frac{\xi^2}{4\tau'})$ evaluated at $s=-\mathrm{i}$ (Doetsch 1961). Consequently, (A 1) becomes

$$\begin{aligned} \theta(\xi, \infty) &= \mathrm{Re} L\left[\frac{\xi}{2\sqrt{\pi}\tau'^{3/2}} \exp\left(-\frac{\xi^2}{4\tau'}\right)\right]\Bigg|_{s=-\mathrm{i}} \\ &= \mathrm{Re}\ \exp(-\xi\sqrt{s})|_{s=-\mathrm{i}} \\ &= \mathrm{Re}\ \exp[-\xi(1-\mathrm{i})/\sqrt{2}] \\ &= \cos(\xi/\sqrt{2}) \exp(-\xi/\sqrt{2}), \end{aligned} \tag{A 2}$$

in agreement with (28) when $Pr=1$.

Similarly, by considering (23) in the form

$$W(\xi, \infty) = \mathrm{Im} \int_0^\infty \exp(\mathrm{i}\tau') \frac{\xi}{2\sqrt{\pi}\tau'^{3/2}} \exp\left(-\frac{\xi^2}{4\tau'}\right) \mathrm{d}\tau',$$

we arrive at $W(\xi, \infty) = \sin(\xi/\sqrt{2}) \exp(-\xi/\sqrt{2})$, which agrees with (29) when $Pr=1$.

Next, consider the case where a plate heat flux is suddenly imposed at $\tau=0$. Rewrite (31) in the limit $\tau \to \infty$ as

$$\theta(\xi, \infty) = \int_0^\infty \frac{\sin\tau'}{\sqrt{\pi}\tau'^{3/2}} \mathrm{d}\tau' \times \int_0^\infty \frac{\xi\cos\tau''}{2\sqrt{\pi}\tau''^{3/2}} \exp\left(-\frac{\xi^2}{4\tau''}\right) \mathrm{d}\tau''. \tag{A 3}$$

Integrate the first integral in (A 3) by parts, and put the result in the form:

$$\theta(\xi, \infty) = 2\mathrm{Re} \int_0^\infty \frac{\exp(\mathrm{i}\tau')}{\sqrt{\pi\tau'}} \mathrm{d}\tau' \times \mathrm{Re} \int_0^\infty \frac{\xi\exp(\mathrm{i}\tau')}{2\sqrt{\pi}\tau''^{3/2}} \exp\left(-\frac{\xi^2}{4\tau''}\right) \mathrm{d}\tau''. \tag{A 4}$$

These integrals can be identified as tabulated Laplace transforms evaluated at $s=-\mathrm{i}$ (Doetsch 1961), and we find

$$\begin{aligned} \theta(\xi, \infty) &= 2\mathrm{Re} L\left(\frac{1}{\sqrt{\pi\tau'}}\right)\Bigg|_{s=-\mathrm{i}} \times \mathrm{Re} L\left[\frac{\xi}{2\sqrt{\pi}\tau''^{3/2}} \exp\left(-\frac{\xi^2}{4\tau''}\right)\right]\Bigg|_{s=-\mathrm{i}} \\ &= 2\mathrm{Re} \frac{1}{\sqrt{s}}\Bigg|_{s=-\mathrm{i}} \times \mathrm{Re}\ \exp(-\xi\sqrt{s})|_{s=-\mathrm{i}} \\ &= 2\mathrm{Re} \frac{\sqrt{2}}{(1-\mathrm{i})} \times \mathrm{Re}\ \exp[-\xi(1-\mathrm{i})/\sqrt{2}] \\ &= \sqrt{2}\cos(\xi/\sqrt{2}) \exp(-\xi/\sqrt{2}). \end{aligned} \tag{A 5}$$

Similarly it can be shown that $W(\xi, \infty) = \sqrt{2}\sin(\xi/\sqrt{2}) \exp(-\xi/\sqrt{2})$. Thus, as $\tau \to \infty$, the unsteady solutions (31) and (34) approach the steady-state solutions (36) and (37) when $Pr=1$.

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