Summer School “**Severe Convective Weather: Theory and Applications**”
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II. Basic Approaches toward Atmospheric Boundary Layer Modeling and Simulation

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Outline

- Governing equations of ABL dynamics and thermodynamics;
- Boussinesq approximation; applicability of equations to atmospheric turbulent flow;
- Reynolds averaging; turbulence closure;
- Concept of LES; subgrid turbulence closure;
- Modeling ABL interactions with underlying surface;
- Parameterization of sheared CBL entrainment in atmospheric models.

Mass conservation for incompressible fluid (continuity equation)

***Vector form*:**

$$\nabla \cdot \mathbf{v} = 0,$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the flow (wind) velocity vector.

***Component form*:**

$$\frac{\partial v_i}{\partial x_i} = 0, \; i=1,\, 2,\, 3,$$

(*Einstein's summation convention is used!*), or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

where $x = x_1,\; y = x_2,\; z = x_3$ are Cartesian coordinates, and

$$u = v_1,\; v = v_2,\; w = v_3$$

are corresponding velocity vector components.

Momentum balance (Navier-Stokes equations)

***Vector form*:**

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{g} - f\mathbf{k} \times \mathbf{v} + \nu \nabla^2 \mathbf{v}$$, **where**

t is **time**, p is **pressure**, ρ is **density**,

$f = 2|\mathbf{\Omega}|\sin\varphi$ is **Coriolis parameter** (φ is **geographic latitude**),

$\mathbf{g} \equiv (g_1, g_2, g_3) = (0, 0, -g)$ is **gravitational acceleration** vector,

ν is **kinematic viscosity** (considered to be constant).

***Component form*:**

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - g\delta_{i3} + \varepsilon_{ij3} f v_j + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$, i=1, 2, 3; j=1, 2, 3.

Heat balance (thermal energy equation)

***Vector form*:**

$$\frac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla \theta = \nu_h \nabla^2 \theta - \frac{1}{c_p \rho} \nabla \cdot \mathbf{F}_r + \frac{1}{c_p \rho} (H_{\mathrm{so}} - H_{\mathrm{si}}),$$ **where**

T is **temperature**, $\theta = T(p_0 / p)^{R/c_p}$ is **potential temperature**,

$\mathbf{F}_r = (F_{r1}, F_{r2}, F_{r3})$ is **radiative heat flux** (vector),

H_{so} and H_{si} are, respectively, **sources and sinks of heat** (in units of energy per unit volume per unit time),

ν_h is **molecular heat diffusivity** (considered to be constant).

***Component form*:**

$$\frac{\partial \theta}{\partial t} + v_i \frac{\partial \theta}{\partial x_i} = \nu_h \frac{\partial^2 \theta}{\partial x_i \partial x_i} - \frac{1}{c_p \rho} \frac{\partial F_{ri}}{\partial x_i} + \frac{1}{c_p \rho} (H_{\mathrm{so}} - H_{\mathrm{si}}),$$ i=1, 2, 3.

ABL governing equations in Boussinesq form

***Equations of motion*:**

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial \pi}{\partial x_i} + \frac{g}{\theta_c}(\theta - \theta_r) + \varepsilon_{ij3} f (v_j - V_{gj}) + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$, **where**

$\pi = (p - p_r)/\rho_c$ is **normalized pressure deviation**,

geostrophic wind vector $\mathbf{V}_g = (V_{g1}, V_{g2})$ is defined through

$$-\frac{1}{\rho_c}\frac{\partial p_r}{\partial x_1} + fV_{g2} = 0, \quad -\frac{1}{\rho_c}\frac{\partial p_r}{\partial x_2} - fV_{g1} = 0,$$

θ_r and p_r are **environmental values** of θ and p,

θ_c and ρ_c **are constant reference values of** θ and ρ,

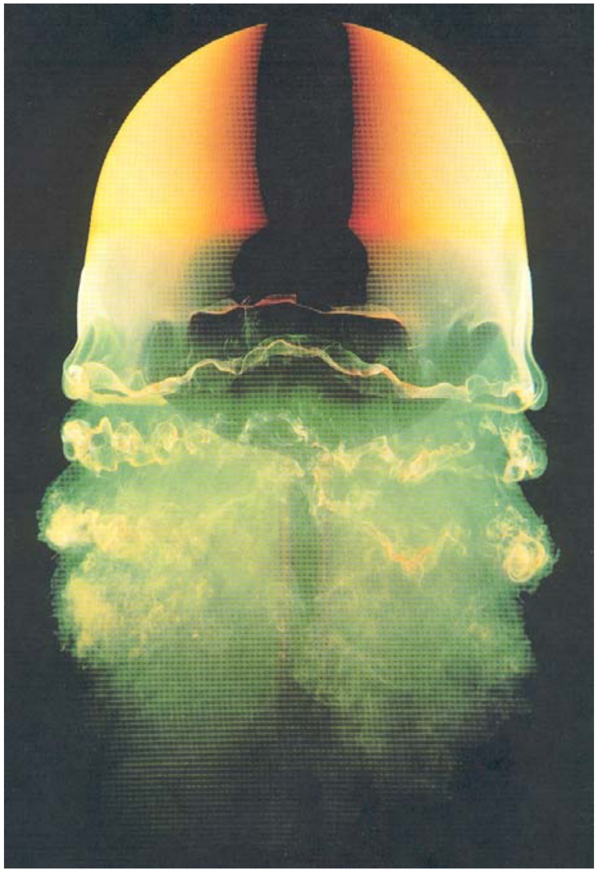
***Heat balance equation*:**

$$\frac{\partial \theta}{\partial t} + v_i \frac{\partial \theta}{\partial x_i} = \nu_h \frac{\partial^2 \theta}{\partial x_i \partial x_i} - \frac{1}{c_p \rho_c}\frac{\partial F_{ri}}{\partial x_i} + \frac{1}{c_p \rho_c}(H_{\text{so}} - H_{\text{si}}).$$

A disturbing fact: ABL flows are turbulent!

Turbulence is a state of moving fluid with contributing motions covering a range of length and velocity scales

Leonardo da Vinci: "Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the ***hair***, the other by the direction of the ***curls***."



Kolmogorov's spectrum

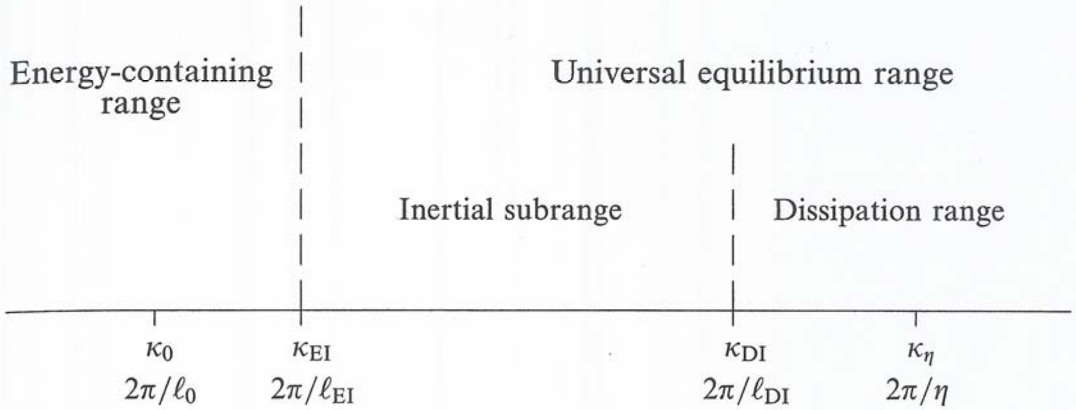


Fig. 6.12. Wavenumbers (on a logarithmic scale) at very high Reynolds number showing the various ranges.

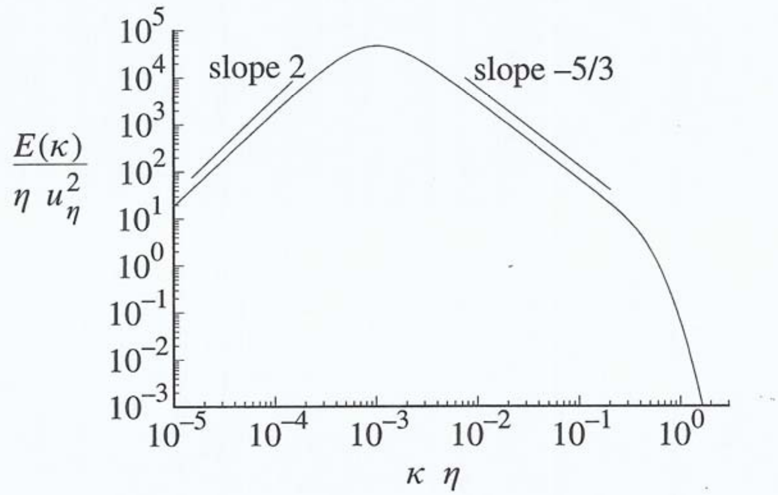


Fig. 6.13. The model spectrum (Eq. (6.246)) for $R_\lambda = 500$ normalized by the Kolmogorov scales.

After Pope (2000)

Applicability of original Boussinesq equations

Solving (numerically!) the Boussinesq equations,

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial \pi}{\partial x_i} + \frac{g}{\theta_c}(\theta - \theta_r) + \varepsilon_{ij3} f (v_j - V_{gj}) + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j}, \qquad \frac{\partial v_i}{\partial x_i},$$

in a domain of size L_D, one can resolve motions in the scale range from $\mathcal{L} \sim L_D$ (with velocity scale $U_{\mathcal{L}}$) down to

$\ell \sim (\nu^3 / \varepsilon)^{1/4} \sim (\nu^3 \mathcal{L} / U_{\mathcal{L}}{}^3)^{1/4}$ **= Kolmogorov turbulence microscale.**

Currently achievable: $\mathcal{L} / \ell \sim 1000$, so $\mathrm{Re} = U_{\mathcal{L}} \mathcal{L} / \nu \sim (\mathcal{L} / \ell)^{4/3} \sim 10^4$.

Resulting turbulence spectrum is much narrower than ABL turbulence spectrum under typical conditions ($\mathrm{Re} \sim 10^7$ to 10^{10})!

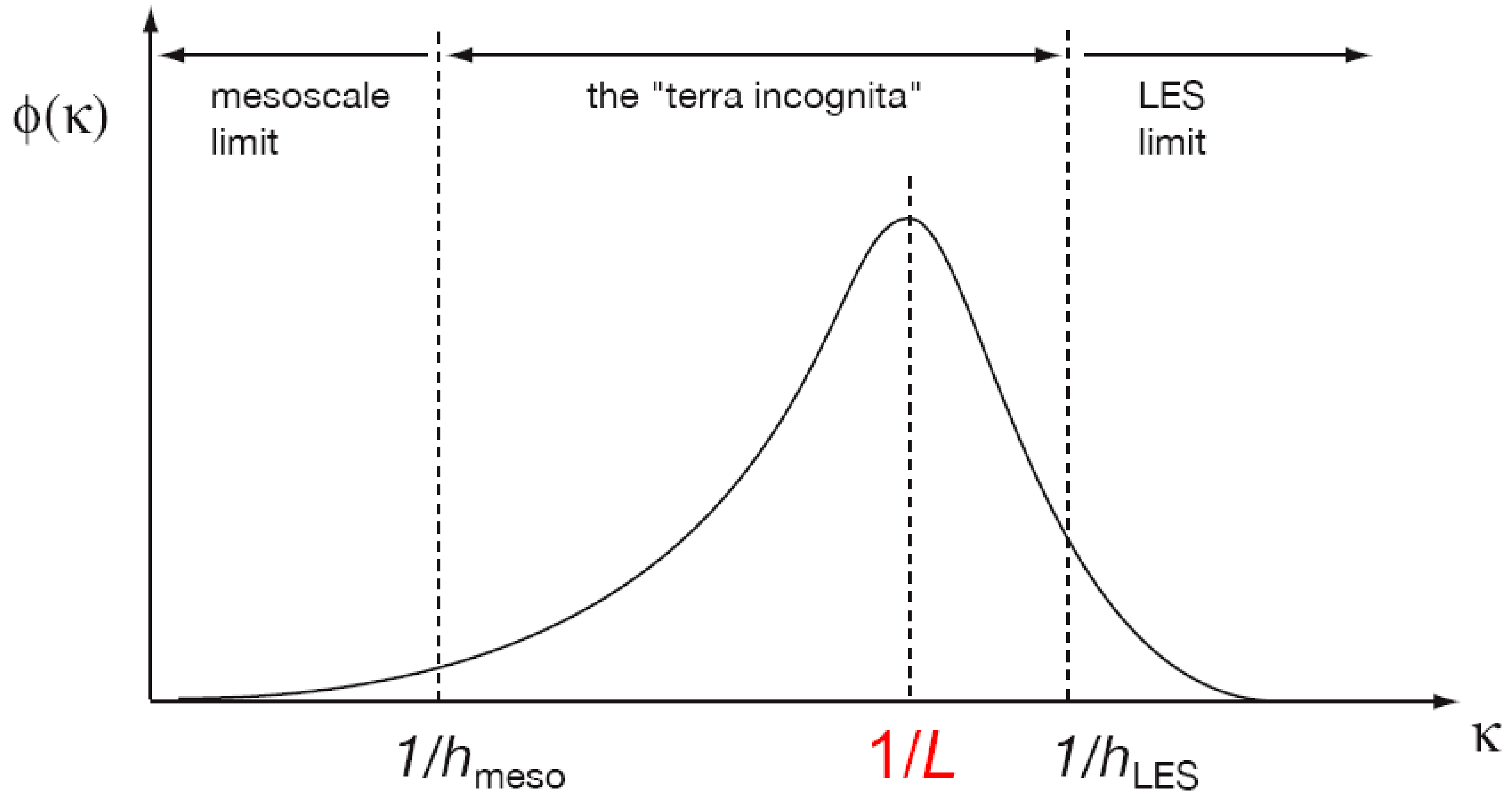
For instance, in order to completely describe an ABL flow with

$$\mathcal{L} = 10^3 \text{ m}, \; U_{\mathcal{L}} = 10 \text{ m s}^{-1}, \text{ and } \nu \sim 10^{-5} \text{ m}^2\text{s}^{-1},$$

one would need to have $\ell \sim 10^{-15/4}\,\mathrm{m}$, that is $L_D / \ell \simeq 10^{11}$!

Filtering/averaging as solution to the problem

(adapted from John Wyngaard)



Schematic of the turbulence spectrum with the peak at 1/*L*.
The averaging/filtering scale *h* may vary.

$h \gg L \gg \ell$ **is the mesoscale limit of ensemble averaging**

$L \gg h \gg \ell$ **is the LES limit of spatial filtering**

Mesoscale modeling (*h*>>*L*): Reynolds averaging

Spatial averaging with implied ensemble averaging rules

Decomposition: $v_i(\mathbf{r},t)=\overline{v}_i(\mathbf{r},t)+v_i'(\mathbf{r},t),\ \theta(\mathbf{r},t)=\overline{\theta}(\mathbf{r},t)+\theta'(\mathbf{r},t),$ **etc.**

Provides in the equations of motion:

$$\frac{\partial(\overline{v}_i+v_i')}{\partial t}+\frac{\partial(\overline{v}_i+v_i')(\overline{v}_j+v_j')}{\partial x_j}=$$

$$-\frac{\partial(\overline{\pi}+\pi')}{\partial x_i}+\frac{g}{\theta_c}(\overline{\theta}+\theta'-\theta_r)\delta_{i3}+\varepsilon_{ij3}f(\overline{v}_j+v_j'-V_{gj})+\nu\frac{\partial^2(\overline{v}_i+v_i')}{\partial x_j\partial x_j}.$$

After averaging (keeping in mind that $\partial\overline{v}_i/\partial x_i=0$):

$$\frac{\partial\overline{v}_i}{\partial t}+\overline{v}_i\frac{\partial\overline{v}_j}{\partial x_j}=-\frac{\partial\overline{\pi}}{\partial x_i}+\frac{g}{\theta_c}(\overline{\theta}-\theta_r)+\varepsilon_{ij3}f(\overline{v}_j-V_{gj})-\frac{\partial\overline{v_i'v_j'}}{\partial x_j}+\nu\frac{\partial^2\overline{v}_i}{\partial x_j\partial x_j},$$

where $\overline{v_i'v_j'}$ are components of the so-called Reynolds stress.

Analogous averaging is applied in the heat balance equation.

Problem of turbulence closure:

how to express $\overline{v_i' v_j'}$ through known flow parameters?

Popular approach based on the Boussinesq analogy:

$$\overline{v_i' v_j'} = \frac{2}{3} e\delta_{ij} - 2k\overline{s}_{ij},\ i=1,\ 2,\ 3;\ j=1,\ 2,\ 3,$$

where

$$\overline{s}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i} \right)$$

is the strain tensor of the mean velocity field,

k **is the eddy viscosity, and**

$$e = \frac{1}{2}\overline{v_i' v_i'} = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}).$$

is the turbulence kinetic energy (TKE) per unit mass.

Now we need a way to prescribe k and e …

The following TKE (e) balance equation may be derived from the original and averaged Navier-Stokes equations:

$$\frac{\partial e}{\partial t}+\bar{v}_j\frac{\partial e}{\partial x_j}=-\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j}+\frac{g}{\theta_c}\overline{v_3'\theta'}-\frac{\partial\overline{(e'+\pi')v_j'}}{\partial x_j}-\nu\overline{\left(\frac{\partial v_i'}{\partial x_j}\right)^2},$$

where $e'\equiv\frac{1}{2}(u'^2+v'^2+w'^2)$**. Further parameterizations:**

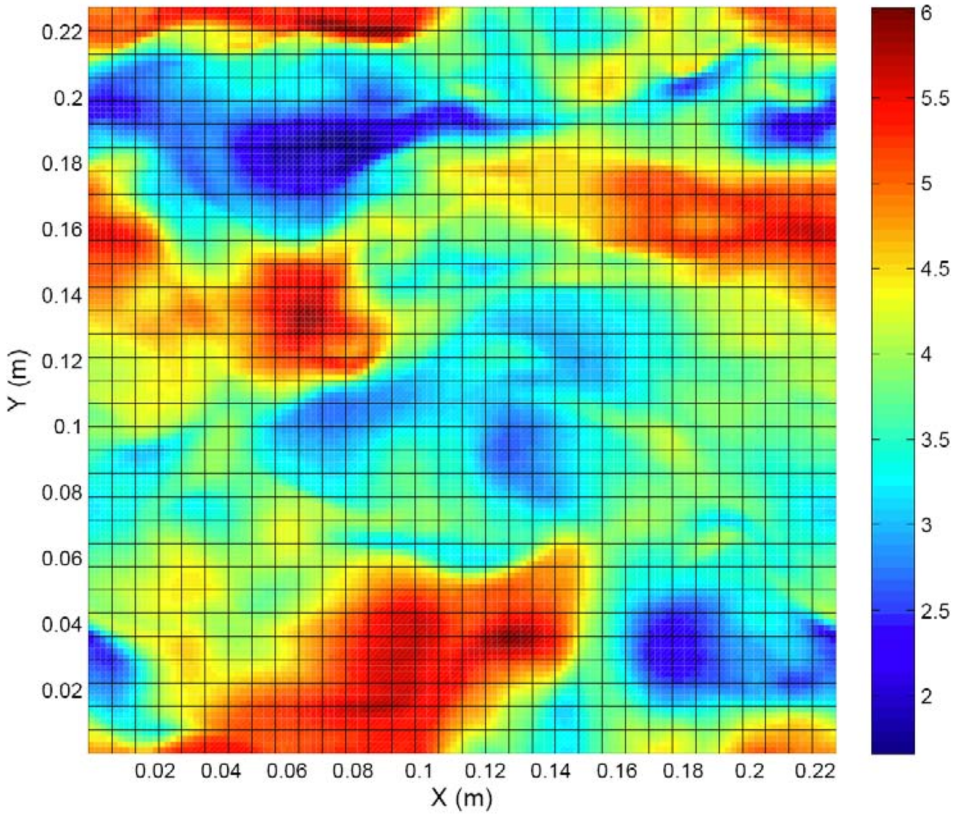
TKE transport: $\frac{\partial\overline{(e'+\pi')v_j'}}{\partial x_j}\propto\frac{\partial}{\partial x_j}\frac{k}{\alpha_e}\frac{\partial e}{\partial x_j}$,

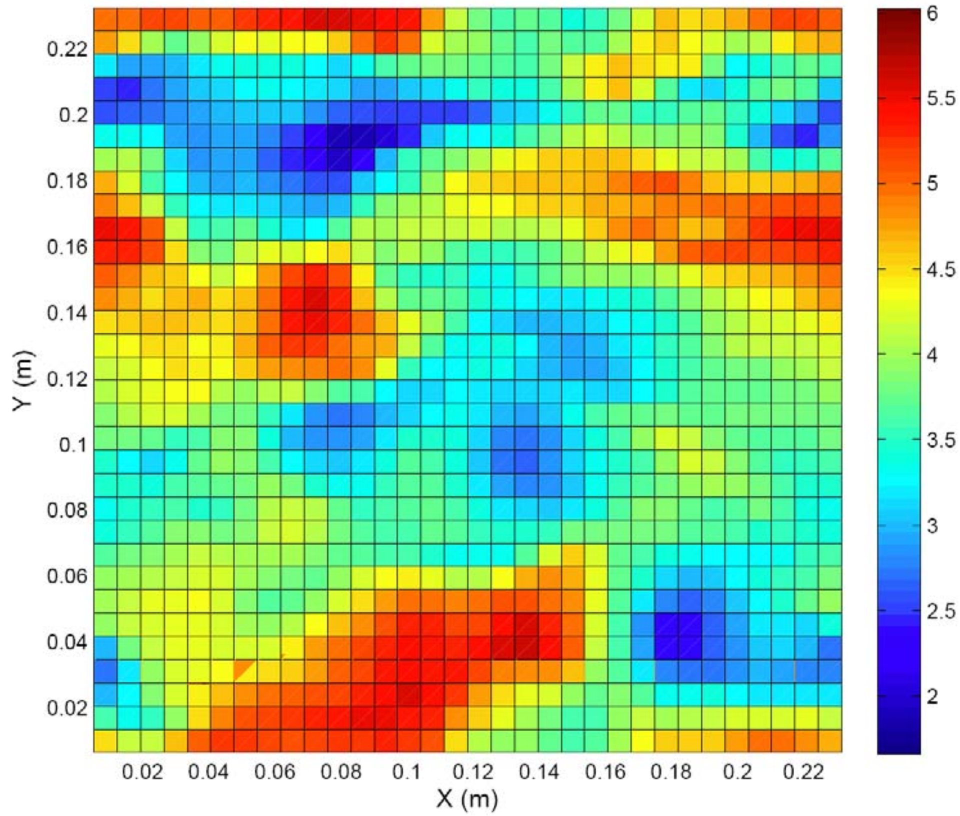
TKE dissipation rate: $\varepsilon=\nu\overline{\left(\frac{\partial v_i'}{\partial x_j}\right)^2}\propto\frac{e^{3/2}}{l_\varepsilon}$ **is.**

Eddy viscosity: $k\propto l_\varepsilon e^{1/2}$**. Prescription of** l_ε **closes the problem!**

Eddy-resolving LES approach (*L*>>*h*)

Concept of spatial filtering





Horizontal velocity component, *u*, in m s^{-1}

Velocity **projected** on LES grid Velocity **filtered** to LES grid

Filtered $(\widetilde{\ldots})$ governing equations in LES

$$\frac{\partial \widetilde{v_i}}{\partial t}+\frac{\partial \widetilde{v_i}\widetilde{v_j}}{\partial x_j}=-\frac{\partial \tilde{\pi}}{\partial x_i}+\frac{g}{\theta_c}(\tilde{\theta}-\theta_r)+\varepsilon_{ij3}f(\tilde{v}_j-V_{gj})+\nu\frac{\partial^2 \widetilde{v_i}}{\partial x_j \partial x_j}-\frac{\partial}{\partial x_j}(\widetilde{v_i v_j}-\widetilde{v_i}\widetilde{v_j}),$$

$$\frac{\partial \widetilde{v_i}}{\partial x_i}=0,$$

$$\frac{\partial \tilde{\theta}}{\partial t}+\frac{\partial \tilde{\theta}\widetilde{u_j}}{\partial x_j}=\kappa\frac{\partial^2 \tilde{\theta}}{\partial x_j \partial x_j}-\frac{\partial}{\partial x_j}(\widetilde{\theta u_j}-\tilde{\theta}\widetilde{u_j}),$$

where $\widetilde{u_i u_j}-\widetilde{u_i}\widetilde{u_j}=-\tau_{ij}^s$ **subgrid kinematic momentum flux,**

$\widetilde{\theta u_j}-\tilde{\theta}\widetilde{u_j}=Q_j^s$ **subgrid kinematic heat flux.**

One would need to specify τ_{ij}^s and Q_j^s through a subgrid closure model!

Subgrid TKE model (Deardorff 1980)

Momentum flux: $$-\tau_{ij}^s = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j = \frac{2}{3} E^s \delta_{ij} - K_m^s \left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right),$$

Buoyancy flux: $$Q_j^s = \widetilde{\theta u_j} - \tilde{\theta} \tilde{u}_j = -K_h^s \frac{\partial \tilde{\theta}}{\partial x_j},$$ **where**

$$K_m^s = C_D l \sqrt{E^s}, \quad C_D = 0.12, \quad K_h^s = K_m^s / \Pr^s, \quad \Pr^s = (1 + 2l/\Delta)^{-1},$$

$l = \Delta$ **if** $\partial \tilde{b} / \partial z \le 0$ **and** $l = \min\left[\Delta,\ 0.5 \sqrt{E^s / (\partial \tilde{b} / \partial z)} \right]$ **if** $\partial \tilde{b} / \partial z \le 0$.

Subgrid TKE balance equation:

$$\frac{\partial E^s}{\partial t} + \frac{\partial \tilde{u}_i E^s}{\partial x_i} = 2 K_m^s \tilde{S}_{ij} \tilde{S}_{ij} + \frac{g}{\theta_c} Q_3^s + \frac{\partial}{\partial x_i} 2 K_m^s \frac{\partial E^s}{\partial x_i} - \varepsilon^s,$$

where the subgrid TKE dissipation rate ε^s is parameterized as

$$\varepsilon^s = f_c(z_m, \Delta) \left[0.19 + 0.51 (l/\Delta) \right] \left(E^s \right)^{3/2} / l.$$

Scale-invariant and scale-dependent subgrid models

(Porté-Agel *et al*. 2000, 2004)

Parameterizations: $K_m^s = (C_{Pm}\Delta)^2 \sqrt{2\widetilde{S}_{ij}\widetilde{S}_{ij}}$ **and** $K_h^s = (C_{Ph}\Delta)^2 \sqrt{2\widetilde{S}_{ij}\widetilde{S}_{ij}}$.

Scale-invariant model: $C_{Pm}{}^2(\Delta) = C_{Pm}{}^2(2\Delta)$, $C_{Ph}{}^2(\Delta) = C_{Ph}{}^2(2\Delta)$,

$$C_{Pm}^2 = \frac{\langle L_{ij} M_{ij} \rangle}{\langle M_{ij} M_{ij} \rangle}, \quad L_{ij} = \overline{\widetilde{u}_i \widetilde{u}_j} - \overline{\widetilde{u}}_i \overline{\widetilde{u}}_j, \quad M_{ij} = 2\Delta^2 \left(\overline{\widetilde{S}_{ij}\sqrt{2\widetilde{S}_{ij}\widetilde{S}_{ij}}} - 4\sqrt{2\overline{\widetilde{S}}_{ij}\overline{\widetilde{S}}_{ij}}\,\overline{\widetilde{S}}_{ij} \right),$$

$$C_{Ph}^2 = \frac{\langle \kappa_{ij} \chi_{ij} \rangle}{\langle \chi_{ij} \chi_{ij} \rangle}, \quad \kappa_{ij} = \overline{\widetilde{u}_i \widetilde{\theta}} - \overline{\widetilde{u}}_i \overline{\widetilde{\theta}}, \quad \chi_{ij} = \Delta^2 \left(\overline{\frac{\partial \widetilde{\theta}}{\partial x_i}\sqrt{2\widetilde{S}_{ij}\widetilde{S}_{ij}}} - 4\sqrt{2\overline{\widetilde{S}}_{ij}\overline{\widetilde{S}}_{ij}}\,\frac{\partial \overline{\widetilde{\theta}}}{\partial x_i} \right),$$

where $\langle \ldots \rangle$ is ***local average*** and $\overline{\ldots}$ is ***filtering*** from Δ to 2Δ.

Scale-dependent model: additional filter 4Δ is implemented

with $\beta = \frac{C_{Pm}{}^2(2\Delta)}{C_{Pm}{}^2(\Delta)} = \frac{C_{Pm}{}^2(4\Delta)}{C_{Pm}{}^2(2\Delta)} \neq 1$ and $\beta_h = \frac{C_{Ph}{}^2(2\Delta)}{C_{Ph}{}^2(\Delta)} = \frac{C_{Ph}{}^2(4\Delta)}{C_{Ph}{}^2(2\Delta)} \neq 1$.

How to relate structural features of averaged/filtered ABL fields near the surface to heat/mass/momentum fluxes?

Commonly used technique is the Monin-Obukhov similarity

Reintroduce (see Lecture I) buoyancy $b = g(\theta_v - \theta_{vr})/\theta_c$.

Scales: ***velocity*** $u_* = (-\overline{u'w'})^{1/2}$, ***temperature*** $\theta_* = -\overline{w'\theta'}/u_*$,
humidity $q_* = -\overline{w'q'}/u_*$, ***buoyancy*** $b_* = -\overline{w'b'}/u_* = \beta\theta_* + 0.61 g q_*$.

Monin-Obukhov (M-O) scaling and universal functions:

$$\frac{\kappa z}{u_*}\frac{\partial \overline{u}}{\partial z} = \kappa \frac{z}{L}\varphi_m'(z/L) \equiv \varphi_m(\zeta), \qquad \frac{\kappa z}{\theta_*}\frac{\partial \overline{\theta}}{\partial z} = \kappa \frac{z}{L}\varphi_h'(z/L) \equiv \varphi_h(\zeta),$$

$$\frac{\kappa z}{q_*}\frac{\partial \overline{q}}{\partial z} = \kappa \frac{z}{L}\varphi_q'(z/L) \equiv \varphi_q(\zeta), \qquad \frac{\kappa z}{b_*}\frac{\partial \overline{b}}{\partial z} = \kappa \frac{z}{L}\varphi_b'(z/L) \equiv \varphi_b(\zeta),$$

M-O length $L = -\dfrac{(-\overline{u'w'})^{3/2}}{\kappa \overline{w'b'}} = \dfrac{u_*^{\,2}}{\kappa b_*}$, $\quad \zeta = z/L, \quad \varphi_h(\zeta) \approx \varphi_q(\zeta) \approx \varphi_b(\zeta)$.

Overbars in the above expressions signify some generic spatial averaging.

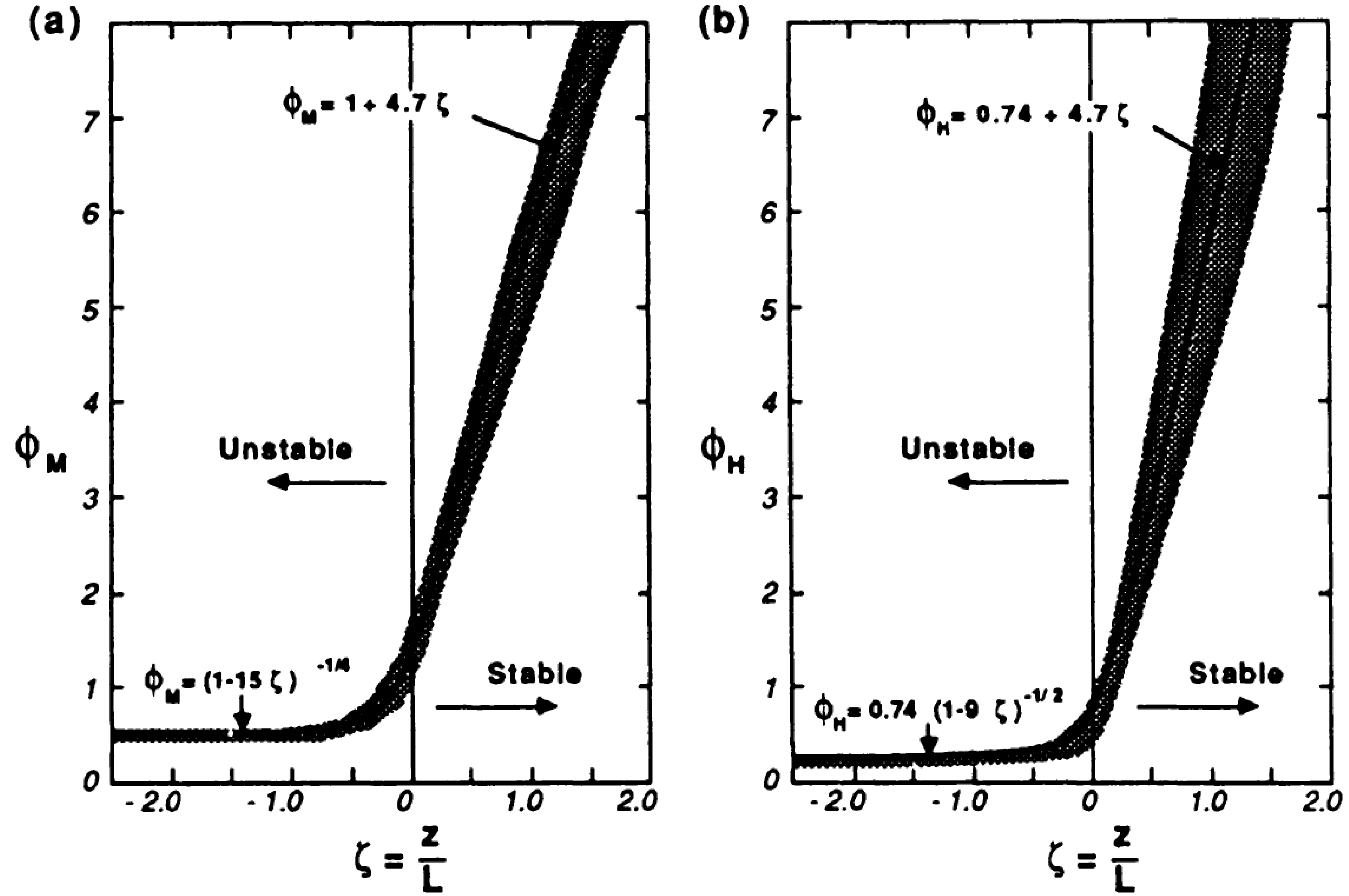


Fig. 9.9 (a) Range of dimensionless wind shear observations in the surface layer, plotted with interpolation formulas. (b) Range of dimensionless temperature gradient observations in the surface layer, plotted with interpolation formulas. After Businger, et al. (1971).

Empirical approximations of M-O similarity functions

Stable conditions ($\zeta = z/L > 0$; $\overline{w'b'} < 0$):

Businger *et al*., $\varphi_m(\zeta) = 1 + 4.7\zeta$, $\varphi_h(\zeta) = 0.74 + 4.7\zeta$, $\kappa = 0.35$.

Dyer *et al*., $\varphi_m(\zeta) = 1 + 5\zeta$, $\varphi_h(\zeta) = 1 + 5\zeta$, $\kappa = 0.41$.

Unstable conditions ($\zeta = z/L < 0$; $\overline{w'b'} > 0$):

Businger *et al*., $\varphi_m(\zeta) = (1 - 15\zeta)^{-1/4}$, $\varphi_h(\zeta) = 0.74(1 - 9\zeta)^{-1/2}$, $\kappa = 0.35$.

Dyer *et al*., $\varphi_m(\zeta) = (1 - 16\zeta)^{-1/4}$, $\varphi_h(\zeta) = (1 - 16\zeta)^{-1/2}$, $\kappa = 0.41$.

Commonly used nowadays are Dyer *et al*. functions with $\kappa = 0.4$.

Neutral conditions ($\zeta = z/L = 0$; $\overline{w'b'} = 0$):

$\varphi_m(0) = 1$, **that is** $\dfrac{\kappa z}{u_*}\dfrac{\partial \overline{u}}{\partial z} = 1$ **or** $\partial \overline{u} / \partial z = u_* / (\kappa z)$, **or** $\overline{u} = (u_* / \kappa) \ln z + C$.

Fixing C so that $\overline{u}(z_0) = 0$, we get the log wind law: $\overline{u} = \dfrac{u_*}{\kappa} \ln \dfrac{z}{z_0}$.

Turbulent diffusivities and Richardson numbers

Turbulent diffusivities in 1-D parallel surface-layer flow:

$$u_*^{\,2} = -\overline{u'w'} = k\frac{\partial \overline{u}}{\partial z} \text{ and } -\overline{w'\theta'} = u_*\theta_* = k_h\frac{\partial \overline{\theta}}{\partial z}.$$

M-O similarity relations for k and k_h:

$$k(z) = \frac{\kappa u_* z}{\varphi_m(\zeta)} = \kappa u_* L\frac{\zeta}{\varphi_m(\zeta)}, \qquad k_h(z) = \frac{\kappa u_* z}{\varphi_h(\zeta)} = \kappa u_* L\frac{\zeta}{\varphi_h(\zeta)}.$$

Flux Richardson number:

$$\mathrm{Ri}_f = \frac{\overline{w'b'}}{\overline{u'w'}(\partial\overline{u}/\partial z)} = \frac{k_h}{k}\mathrm{Ri} = \frac{\mathrm{Ri}}{\mathrm{Pr}_t}.$$

Gradient Richardson number:

$$\mathrm{Ri} = \frac{\partial\overline{b}/\partial z}{\left(\partial\overline{u}/\partial z\right)^2} = \frac{k}{k_h}\mathrm{Ri}_f = \mathrm{Pr}_t\mathrm{Ri}_f.$$

$$\mathrm{Pr}_t = \varphi_h/\varphi_m = k/k_h = \mathrm{Ri}/\mathrm{Ri}_f, \qquad \mathrm{Ri} = \frac{\varphi_h}{\varphi_m^{\,2}}\zeta = \frac{\mathrm{Pr}_t}{\varphi_m}\zeta, \qquad \mathrm{Ri}_f = \frac{1}{\varphi_m}\zeta = \frac{\mathrm{Pr}_t}{\varphi_h}\zeta.$$

Effects of stability on surface-layer parameters

Stable conditions ($\zeta = z/L > 0$; $\overline{w'b'} < 0$) **after Dyer *et al.*:**

$$\mathrm{Ri} = \mathrm{Ri}_f = \zeta/(1+5\zeta) \geq 0, \qquad \zeta = \mathrm{Ri}/(1-5\mathrm{Ri}).$$

Therefore, Ri=**0.2** ($\zeta \to \infty, L \to 0$) **is interpreted as a critical** Ri.

Unstable conditions ($\zeta = z/L < 0$; $\overline{w'b'} > 0$) **after Dyer *et al.*:**

$$\mathrm{Ri} = \zeta \leq 0, \qquad \mathrm{Ri}_f = \zeta(1-16\zeta)^{1/4} \leq 0.$$

Integrating $\dfrac{\kappa z}{u_*}\dfrac{\partial u}{\partial z} = \varphi_m(\zeta)$ **and** $\dfrac{\kappa z}{\theta_*}\dfrac{\partial \theta}{\partial z} = \varphi_h(\zeta)$**, one obtains**

$$u(z) = \frac{u_*}{\kappa}\left[\ln\frac{z}{z_0} - \Psi_m\left(\frac{z}{L}\right)\right] \text{ and } \theta(z) = \theta_s + \frac{\theta_*}{\kappa}\left[\ln\frac{z}{z_{0\theta}} - \Psi_h\left(\frac{z}{L}\right)\right],$$

where θ_s **is surface potential temperature,**

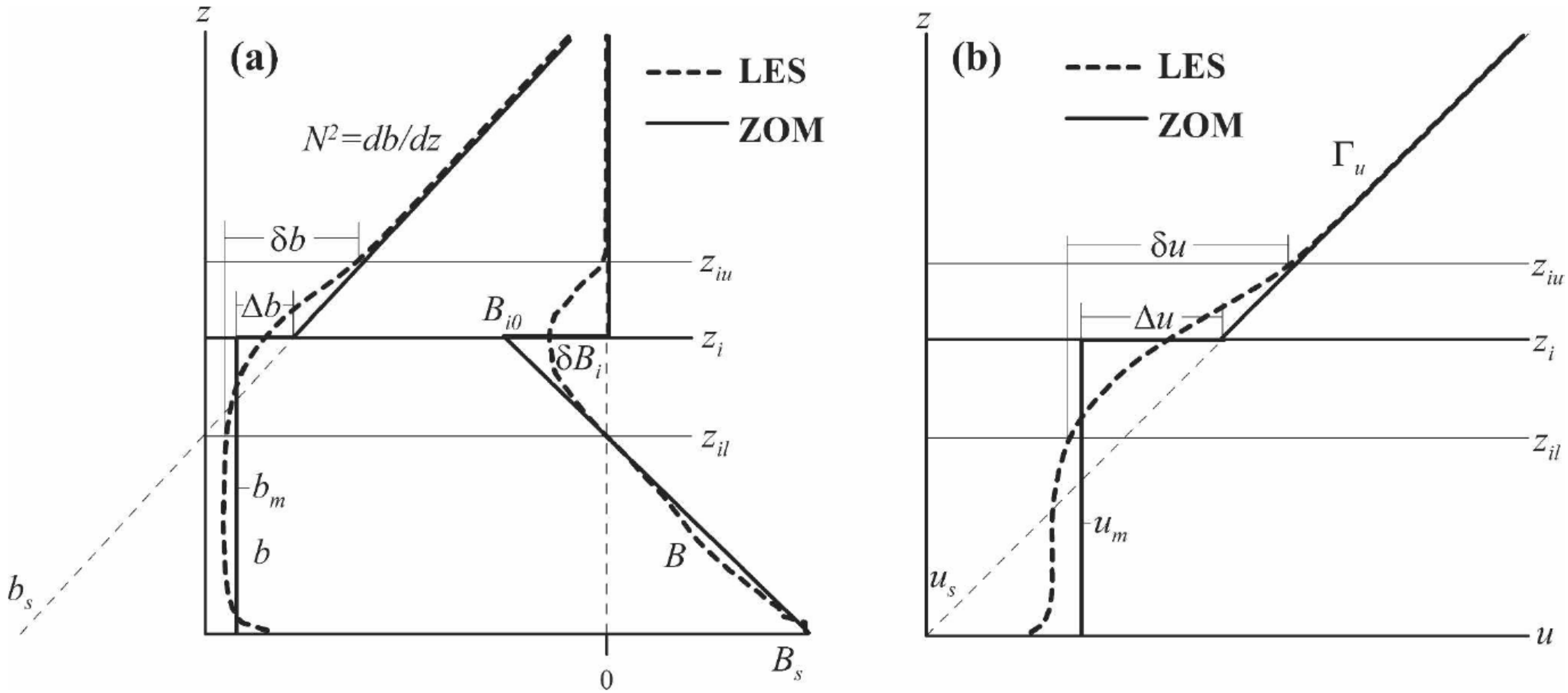
z_0 **is roughness length for velocity (momentum),**

$z_{0\theta}$ **is roughness length for temperature (heat).**

Parameterization of CBL in atmospheric models

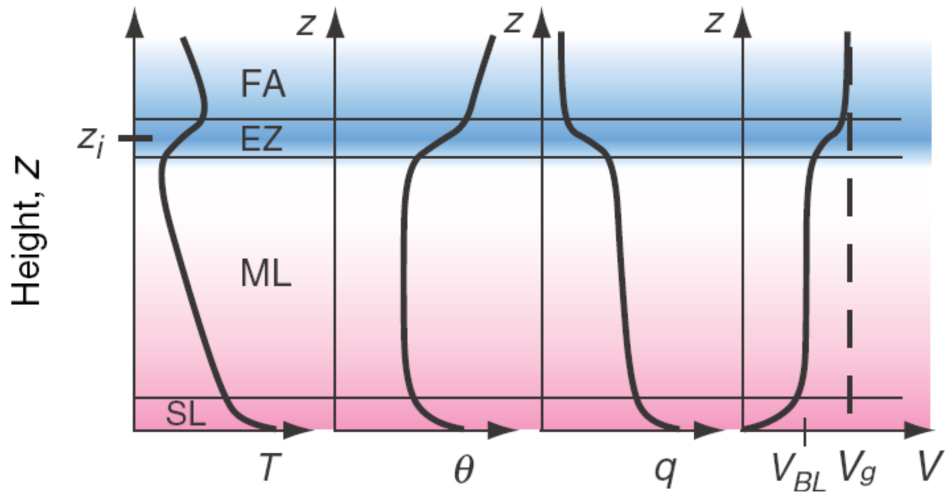
Zero-order bulk model (ZOM) of CBL

after Conzemius and Fedorovich (2006)



$$b_m = \frac{1}{z_i}\int_0^{z_i} b\,dz, \quad u_m = \frac{1}{z_i}\int_0^{z_i} u\,dz, \quad \upsilon_m = \frac{1}{z_i}\int_0^{z_i} \upsilon\,dz.$$

CBL ZOM equations



$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{\partial \overline{\theta' w'}}{\partial z} \qquad \frac{\partial \overline{q}}{\partial t} = -\frac{\partial \overline{q' w'}}{\partial z}$$

$$\frac{\partial \overline{u}}{\partial t} = f(\overline{v} - V_g) - \frac{\partial \overline{u' w'}}{\partial z}, \qquad \frac{\partial \overline{v}}{\partial t} = -f(\overline{u} - U_g) - \frac{\partial \overline{v' w'}}{\partial z},$$

Combine θ and q equations, and integrate over z to get

$$z_i \frac{db_m}{dt} = \Delta b \frac{dz_i}{dt} + B_s; \qquad z_i \frac{du_m}{dt} = \Delta u \frac{dz_i}{dt} + z_i \left[f\left(v_m - V_{g0}\right) - \frac{z_i}{2} f \Gamma_v \right] + \tau_{xs};$$

$$z_i \frac{dv_m}{dt} = \Delta v \frac{dz_i}{dt} - z_i \left[f\left(u_m - U_{g0}\right) - \frac{z_i}{2} f \Gamma_u \right] + \tau_{ys}.$$

TKE budget in the ZOM of CBL

TKE balance in a horizontally homogeneous BL flow:

$$\frac{\partial e}{\partial t} = -\overline{u'w'}\frac{\partial \overline{u}}{\partial z} - \overline{v'w'}\frac{\partial \overline{v}}{\partial z} + \overline{w'b'} - \frac{\partial \overline{(e'+\pi')w'}}{\partial z} - \varepsilon .$$

Integrating over z using ZOM representation of CBL structure:

$$\left[e_m - \frac{1}{2}\left(\Delta u^2 + \Delta v^2 - \Delta b z_i \right) \right]\frac{dz_i}{dt} + \frac{de_m}{dt} z_i = -u_m \tau_{xs} - v_m \tau_{ys} + \frac{B_s z_i}{2} - \Phi_i - \varepsilon_m z_i ,$$

where $e_m = \frac{1}{z_i}\int_0^{z_i} e\,dz$, $\varepsilon_m = \frac{1}{z_i}\int_0^{z_i} \varepsilon\,dz$, **and** $\Phi_i = \overline{(e'+\pi')w'}_{z_i}$.

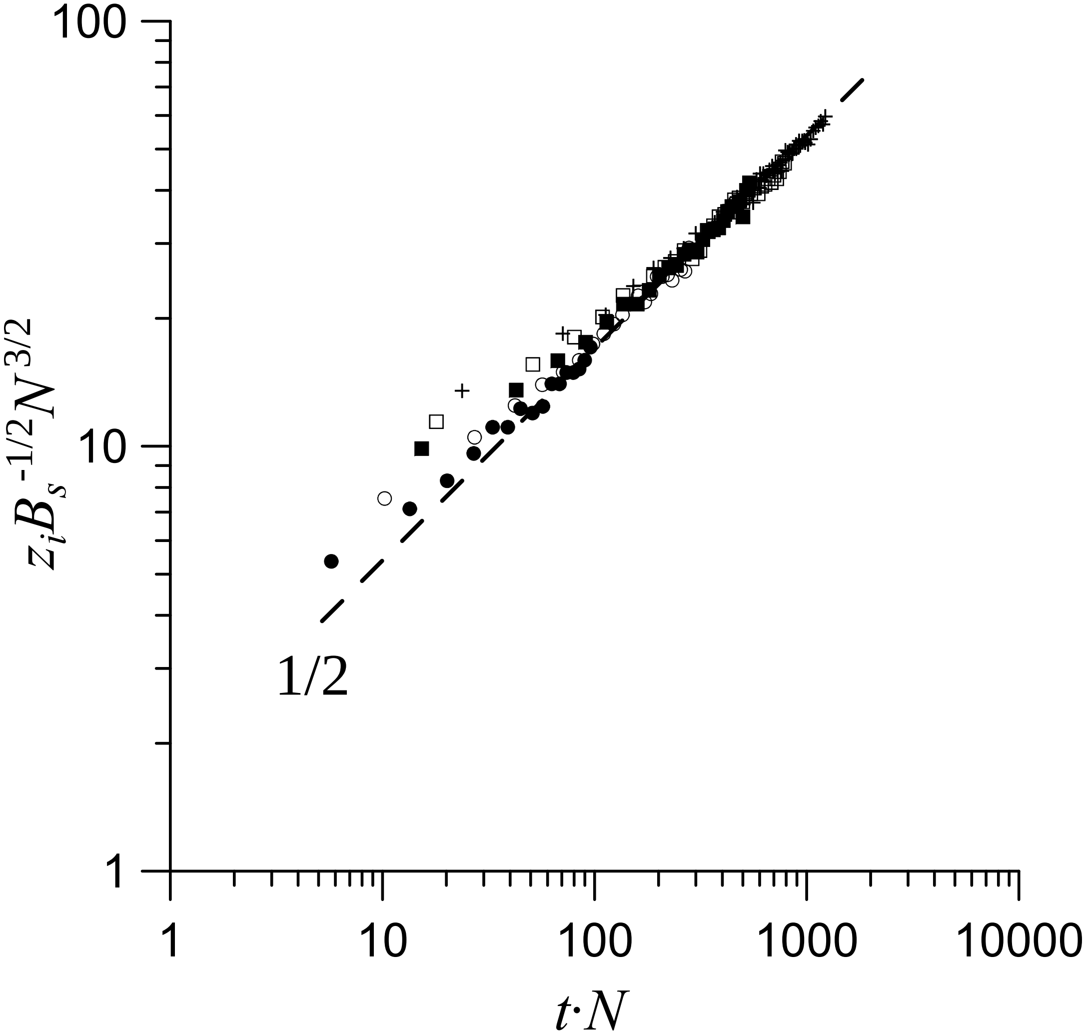
For equilibrium, shear-free entrainment the problem reduces to:

$$\frac{d}{dt}\left(\frac{N^2 z_i^2}{2} - \Delta b z_i \right) = B_s , \qquad \frac{\Delta b (dz_i / dt)}{B_s} = C_1 ,$$

and can be solved analytically:

$$\hat{z}_i = [2(1+2C_1)\hat{t}\,]^{1/2}, \quad \Delta\hat{b} = C_1[2\hat{t}\,/\,(1+2C_1)]^{1/2},$$

where $\hat{z}_i = z_i B_s^{-1/2} N^{3/2}$, $\Delta\hat{b} = \Delta b B_s^{-1/2} N^{-1/2}$ **and** $\hat{t} = tN$.



Dimensionless z_i from LES as a function of dimensionless time. Different symbols correspond to N=0.008s^{-1} (●), N=0.011s^{-1} (○), N=0.014s^{-1} (■), N=0.016s^{-1} (□), N=0.018s^{-1} (+). Dashed line presents the ZOM equilibrium solution from Fedorovich et al. (2004).

Courtesy of Larry Mahrt