Review of second order linear constant coefficient ordinary differential equations (ODEs)

METR3123, Alan Shapiro, Instructor

The general **2nd order linear constant coefficient ODE** is

$$\frac{d^2u}{dt^2}+A\frac{du}{dt}+Bu=C\,, \qquad (1)$$

where A, B, and C are coefficients (considered here as constants), u is the dependent variable, and t is the independent variable. Often we'll think of t as time. But can have space (x, y, or z) in place of t, and v (or T or p or whatever) in place of u; the equation is basically the same, no matter what you call the variables.

The **general solution** of (1) is the solution that **includes all possible solutions** of the equation. From the theory of ODEs, the **general solution must contain two arbitrary constants**.

Often we seek a **particular solution** of (1), that is, the solution of (1) that **satisfies initial conditions**: u and du/dt are specified at $t = 0$ (or one can impose conditions at two points, e.g., at $t = 0$ and at $t = 1$ hr). Different initial conditions lead to different solutions. If we work with x, y, or z instead of t, we discuss **boundary conditions** instead of initial conditions, but the idea is the same.

In practice, to find a particular solution, you first need to find the general solution, and then adjust the two arbitrary constants in it so that the initial conditions are satisfied.

Equation (1) includes a term that does **not** have the dependent variable in it (term C has no u in it). We call such an equation "nonhomogeneous" or "inhomogeneous". So (1) is a 2nd order linear constant coefficient **inhomogeneous** ODE. The version of (1) without the inhomogeneous term,

$$\frac{d^2u}{dt^2}+A\frac{du}{dt}+Bu=0, \tag{2}$$

is a 2nd order linear constant coefficient **homogeneous** ODE.

We'll review the solution procedure for (2) and then review the procedure for (1). Actually, the basic procedure is illustrated more simply by working with an equation even simpler than (2):

$$\frac{d^2u}{dt^2}+Bu=0, \tag{3}$$

Yeah, lets work with this (3) instead of the messier (2).

We've seen that the solution of linear homogeneous 1st order ODEs (e.g., $du/dt+Bu=0$) is an exponential. It turns out that exponentials also work in the 2nd order case, but these may be complex and thus lead to sines and cosines (via Euler's formula). So consider the trial solution (try it and see if it works) in the form

$$u=e^{\lambda t}, \tag{4}$$

where λ is an as-yet-unknown constant. Plug (4) into (3),

$$\frac{d^2e^{\lambda t}}{dt^2}+Be^{\lambda t}=0, \qquad \text{use fact that } d(e^{\lambda t})/dt=\lambda\, e^{\lambda t} \text{ (twice)}$$

$$\therefore \quad \lambda^2e^{\lambda t}+Be^{\lambda t}=0 \qquad \text{divide by common factor of } e^{\lambda t}$$

$\lambda^2 + B = 0$

$\lambda^2 = -B$ take square root

$\lambda = \pm\sqrt{-B}$

Taking the square root gives us two possible λ. We need **both** in the general solution. The general solution is a **linear combination** of the two possible trial solutions:

$$u(t) = c_1 e^{\sqrt{-B}\,t} + c_2 e^{-\sqrt{-B}\,t}, \qquad (5)$$

where c_1 and c_2 are arbitrary constants.

If B is negative then $-B$ is positive and $\sqrt{-B}$ is a real number. When we impose the initial conditions in (5) to obtain c_1 and c_2, we find that c_1 and c_2 are real.

If B is positive then $-B$ is negative and $\sqrt{-B} = i\sqrt{B}$ is imaginary. The exponentials in (5) are thus of the forms $e^{i\sqrt{B}\,t}$ and $e^{-i\sqrt{B}\,t}$, which are related to sines and cosines via Euler's formula:

$$e^{i\sqrt{B}\,t} = \cos\sqrt{B}t + i\sin\sqrt{B}t, \quad e^{-i\sqrt{B}\,t} = \cos\sqrt{B}t - i\sin\sqrt{B}t. \quad (6)$$

These exponentials are complex. It turns out that for u to be real, the c_1 and c_2 must also be complex. To see what's going on (and see why it's not a problem), consider the following example:

Example: Inertial motion in the atmosphere

Consider the horizontal equations of motion for an air parcel moving under the influence of the Coriolis force (only),

$$\frac{du}{dt} = fv, \tag{7}$$

$$\frac{dv}{dt} = -fu, \tag{8}$$

where *f* is treated as a constant. Taking d/dt of (8) yields

$$\frac{d^2v}{dt^2} = -f\boxed{\frac{du}{dt}}. \tag{9}$$

Into (9) plunk in *du*/*dt* from (7). We thus obtain the 2$^{\text{nd}}$ order linear constant coefficient homogeneous ODE:

$$\frac{d^2v}{dt^2} = -f^2 v. \tag{10}$$

Seek solutions of the form $v = e^{\lambda t}$. Applying it in (10) yields

$\lambda^2 e^{\lambda t} = -f^2 e^{\lambda t}.$ Divide by $e^{\lambda t}$

$\lambda^2 = -f^2.$ Take the square root

$\lambda = \pm if$

So the general solution for *v* is of the form

$$v(t) = c_1 e^{ift} + c_2 e^{-ift}. \tag{11}$$

Apply Euler's formula to each of the exponentials in (11), get

$$v(t) = c_1\left(\cos ft + i\sin ft\right) + c_2\left(\cos ft - i\sin ft\right). \tag{12}$$

Group together cosines with cosines, sines with sines:

$$v(t) = \underbrace{\boxed{(c_1 + c_2)}}_{d_1}\cos ft + \underbrace{\boxed{i(c_1 - c_2)}}_{d_2}\sin ft. \tag{13}$$

We don't know what c_1 or c_2 are. They're just "some constants". So if you add them together you get "some other constant". Call it d_1. Similarly let d_2 be the name for $i(c_1 - c_2)$. So (13) becomes:

$$v(t) = d_1 \cos ft + d_2 \sin ft. \tag{14}$$

No need to work with c_1 or c_2 anymore. Just work with d_1 and d_2. Infer them from the initial conditions. For example, suppose

$$v(0) = 10\,\mathrm{m\,s^{-1}}, \tag{15a}$$

and

$$\left.\frac{dv}{dt}\right|_{t=0} = 0\,\mathrm{m\,s^{-2}} \text{ [from (8) it means } u(0) = 0\,\mathrm{m\,s^{-1}}] \tag{15b}$$

Setting $t = 0$ in (14) and using (15a) yields:

$$\underbrace{\boxed{v(0)}}_{10\,\mathrm{m\,s^{-1}}} = d_1 \underbrace{\boxed{\cos 0}}_{1} + d_2 \underbrace{\boxed{\sin 0}}_{0} = d_1 \qquad \therefore \boxed{d_1 = 10\mathrm{m\,s^{-1}}}$$

Now take d/dt of (14), and **then** set t = 0 (and use (15b):

$$\frac{dv}{dt} = -d_1 f \sin ft + d_2 f \cos ft$$

$$\underbrace{\boxed{\left.\frac{dv}{dt}\right|_{t=0}}}_{0} = -d_1 f \underbrace{\boxed{\sin 0}}_{0} + d_2 f \underbrace{\boxed{\cos 0}}_{1} = d_2 f \qquad \therefore \boxed{d_2 = 0}$$

Applying these d_1 and d_2 values in the general solution (14) yields the solution of our problem:

$$\boxed{v(t) = 10\mathrm{m\,s^{-1}} \cos ft}$$

____________________________________ end of example.

Now consider the inhomogeneous ODE,

$$\frac{d^2u}{dt^2} + Bu = C, \tag{16}$$

We've already obtained the general solution of the **homogeneous version** of it [(3) is the homogeneous version of it, and (5) was the general solution of (3)]. For brevity, we refer to that solution as the homogeneous solution u_h. So (5) says that

$$u_h(t) = c_1 e^{\sqrt{-B}\,t} + c_2 e^{-\sqrt{-B}\,t}$$

To get the general solution of (16), find **any** particular solution u_p of (16) and then **add** it to the homogeneous solution u_h of (16).

In the more general case where *C* is time dependent, one can find a particular solution using variation of parameters. But in this review handout, we're treating *C* as constant so there's no need for variation of parameters. In this case the particular solution is just some constant. Plug $u_p = const$ into (16). Get

$$\boxed{\frac{d^2}{dt^2}(const)} + B(const) = C\,.$$

The first term is 0 since the derivative of a constant is 0. Now, you can tell from inspection that $const = C\,/\,B$.

So the general solution of (16) is:

$$u = u_p + u_h$$

$$= \frac{C}{B} + c_1 e^{\sqrt{-B}\,t} + c_2 e^{-\sqrt{-B}\,t}$$

Qualitative behavior of solution of (3) for B > 0 and B < 0. Done on board. Time permitting.