LECTURE 41

Group Velocity

A single wave (e.g., Rossby) rarely occurs. Usually there's a jumble of such waves. But the sum of the waves is still a solution of the **linearized** governing equations.

Consider **2 waves** of **equal amplitude** but **slightly different wavelength and frequency** moving in same direction:

$$\omega_1 = \bar{\omega} + \Delta\omega, \qquad k_1 = \bar{k} + \Delta k,$$
$$\omega_2 = \bar{\omega} - \Delta\omega, \qquad k_2 = \bar{k} - \Delta k$$

assume $\left|\frac{\Delta\omega}{\bar{\omega}}\right| << 1, \qquad \left|\frac{\Delta k}{\bar{k}}\right| << 1.$

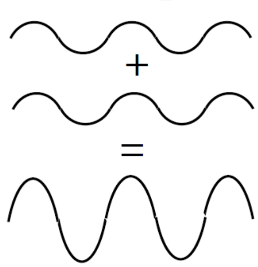
Because of the dispersion relation, $\Delta\omega$ is related to Δk.

Mean frequency is: $\frac{\omega_1 + \omega_2}{2} = \frac{\bar{\omega} + \Delta\omega + \bar{\omega} - \Delta\omega}{2} = \frac{2\bar{\omega}}{2} = \bar{\omega}$

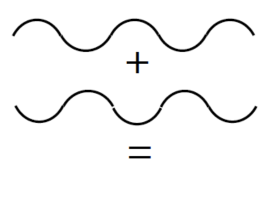
Mean wavenumber is: $\frac{k_1 + k_2}{2} = \frac{\bar{k} + \Delta k + \bar{k} - \Delta k}{2} = \frac{2\bar{k}}{2} = \bar{k}$

Where the waves are in phase (or nearly so), they combine to form a wave of twice amplitude. Where they're out of phase, they kill each other off:

2 waves in phase:



2 waves out of phase:



More specifically, consider a wave property (e.g., ψ) for sum of two waves.

$$\psi = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t).$$

Plug in the expressions for $\omega_1,\ \omega_2,\ k_1, k_2$. **Careful with signs**!

$$\psi = A\cos(\bar{k}\,x + \Delta k\,x - \bar{\omega}t - \Delta\omega\,t) + A\cos(\bar{k}\,x - \Delta k\,x - \bar{\omega}t + \Delta\omega\,t).$$

Put mean wave parameters together. Put deviations together. **Careful with signs**!

$$\psi = A\cos\left[\bar{k}\,x - \bar{\omega}t + (\Delta k\,x - \Delta\omega\,t)\right] + A\cos\left[\bar{k}\,x - \bar{\omega}t - \left(\Delta k\,x - \Delta\omega\,t\right)\right].$$

think of: cos(a + b)] cos(a - b)

Use addition and subtraction formulas for cosines.

$$\psi = A\cos(\bar{k}\,x - \bar{\omega}t)\,\cos(\Delta k\,x - \Delta\omega\,t)\ \boxed{- A\sin(\bar{k}\,x - \bar{\omega}t)\,\sin(\Delta k\,x - \Delta\omega\,t)}$$

cancellation

$$+ A\cos(\bar{k}\,x - \bar{\omega}t)\,\cos(\Delta k\,x - \Delta\omega\,t)\ \boxed{+ A\sin(\bar{k}\,x - \bar{\omega}t)\,\sin(\Delta k\,x - \Delta\omega\,t)}$$

$$= 2A\cos(\bar{k}\,x - \bar{\omega}t)\cos(\Delta k\,x - \Delta\omega\,t)$$

$$\therefore\quad \psi = A_{eff}\ \boxed{\cos(\bar{k}\,x - \bar{\omega}t)} \quad \text{where} \quad A_{eff} = 2A\cos(\Delta k\,x - \Delta\omega\,t)$$

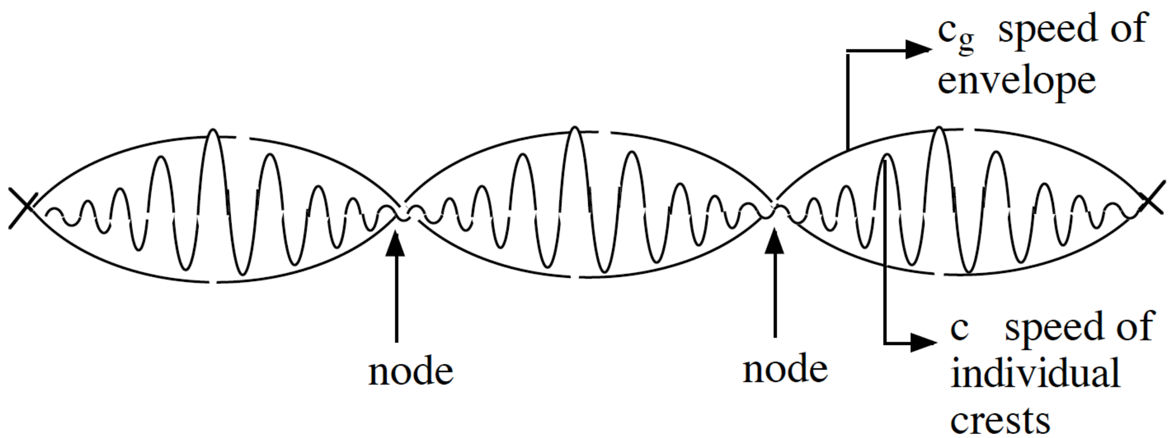
A_{eff} ↓ effective amplitude; $\cos(\bar{k}\,x - \bar{\omega}t)$ ↓ carrier wave (mean wave)

The **effective amplitude** A_{eff} **is itself a wave** with a **small wavenumber** Δk (large wavelength, $\dfrac{2\pi}{\Delta k}$) that propagates with frequency $\Delta\omega$. So speed of propagation of A_{eff} is $\dfrac{\Delta\omega}{\Delta k}$.

In limit of $\Delta k \to 0$, $\dfrac{\Delta\omega}{\Delta k} \to \dfrac{d\omega}{dk}$. Define the **group velocity** $c_G \equiv \dfrac{d\omega}{dk}$.

Phase speed c is speed of individual crests/troughs.

Group velocity c_G is speed of **envelope** of the crests/troughs:



Energy is trapped between nodes. Therefore, energy propagates at speed of nodes (speed of envelope), i.e., with group velocity c_G, not phase speed c.

Notion of group velocity is applicable to any type of wave. Just use the appropriate dispersion relation in the calculation of c_G.

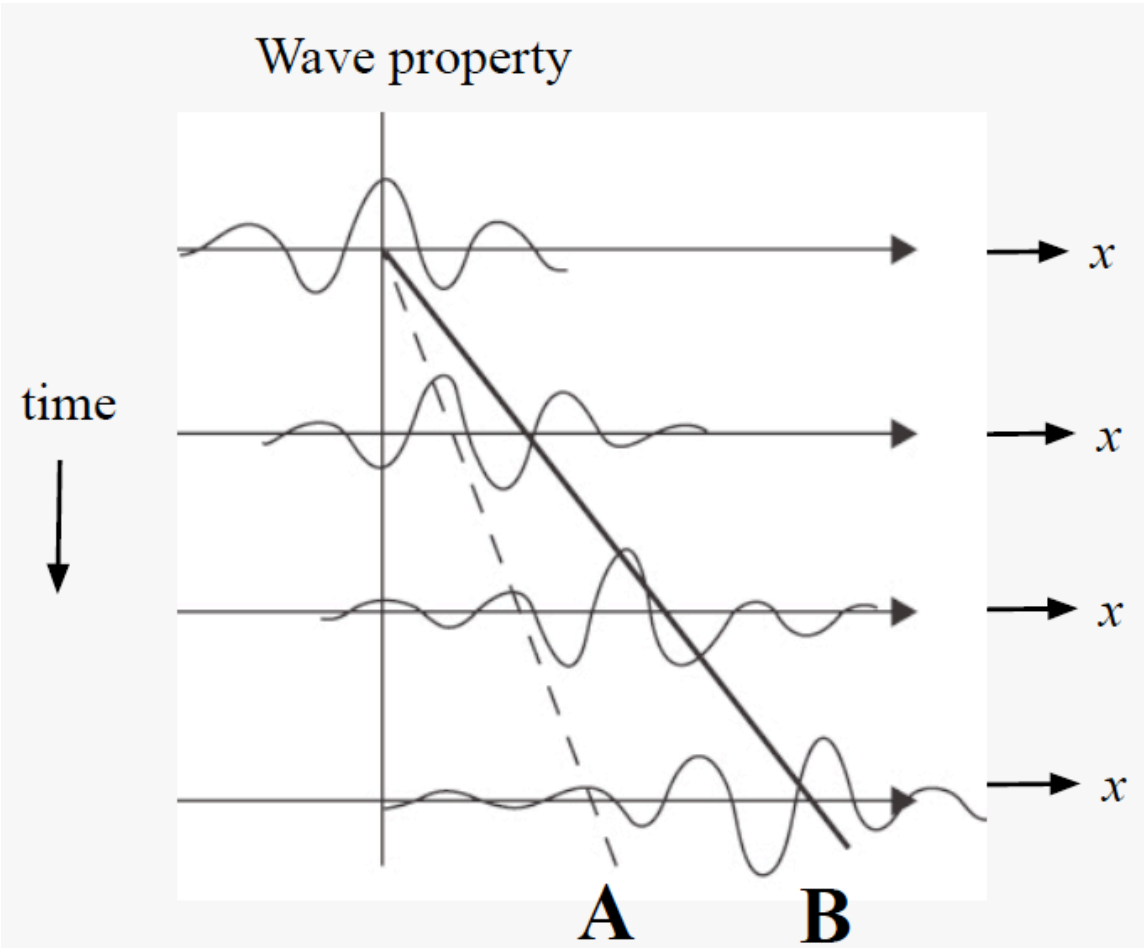
Group velocity for sound waves: $c_G = \dfrac{d\omega}{dk} = \dfrac{d}{dk}\left(Uk \pm k\sqrt{\gamma RT}\right) = U \pm \sqrt{\gamma RT} = c$.

Actually, $c_G = c$ for any non-dispersive wave (prove it!).

Group velocity for Rossby waves: $c_G = \dfrac{d\omega}{dk} = \dfrac{d}{dk}\left(Uk - \dfrac{\beta}{k}\right) = U + \dfrac{\beta}{k^2} > 0$.

So energy propagates eastward! So individual crests can move westward or eastward (depending on sign of $c = U - \dfrac{\beta}{k^2}$), while the envelope and energy move eastward.

Schematic propagation of a Rossby wave group. Dashed line tracks location of a single wave crest that roughly coincided with the peak of the envelope of the disturbance at the initial time. The crest slowly moves eastward with phase speed c. Heavy line tracks peak of the envelope of the disturbance. The envelope moves eastward more quickly, with group velocity c_G (> c).



Extrapolating the position of that single crest from the early time to the later time using the observed phase speed puts the crest at A. Due to "downstream development" (manifestation of energy propagating with group velocity rather than the phase velocity), the peak of the envelope of the disturbance is found at B. The wave that was such a big shot early on has been left behind.