Lecture 10

Kinematics of vorticity and circulation

Vorticity

$\vec{\omega} \equiv \nabla \times \vec{U}$ is vorticity vector.

Careful with notation:
w is vertical velocity
ω is $\frac{Dp}{Dt}$
$\vec{\omega}$ is vorticity

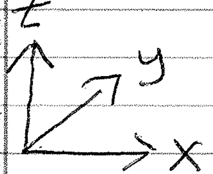
$$\vec{\omega} = \nabla \times \vec{U} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \underbrace{\hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}_{\text{x-comp vorticity}} + \underbrace{\hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}_{\text{y-comp vorticity}} + \underbrace{\hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\text{z-comp vorticity}}$$

$\zeta \equiv \hat{k} \cdot \vec{\omega} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is vertical vorticity [or vertical vorticity component]

Indicator of vorticity: put a pinwheel in flow and see if it rotates.

e.g. <u>constant wind</u>: $u = 10\ m/s$
$v = 0$
$w = 0$



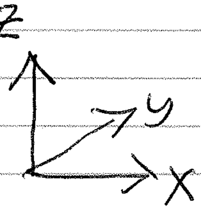
+ pinwheel doesn't rotate

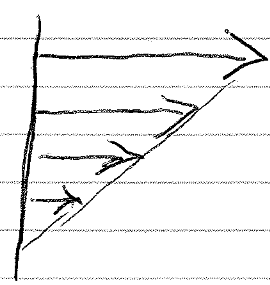
$$\vec{\omega} = \hat{i}\left(\underset{\downarrow \atop 0}{\frac{\partial w}{\partial y}} - \underset{\downarrow \atop 0}{\frac{\partial v}{\partial z}}\right) + \hat{j}\left(\underset{\downarrow \atop 0}{\frac{\partial u}{\partial z}} - \underset{\downarrow \atop 0}{\frac{\partial w}{\partial x}}\right) + \hat{k}\left(\underset{\downarrow \atop 0}{\frac{\partial v}{\partial x}} - \underset{\downarrow \atop 0}{\frac{\partial u}{\partial y}}\right)$$

all component are 0, so $\vec{\omega} = 0$

e.g. <u>unidirectional wind shear</u>: ~~γz~~

$u = \gamma z$ where $\gamma > 0$ is a constant
$v = 0$
$w = 0$





pinwheel rotates about the y ($\hat{j}$) axis.

$$\vec{\omega} = \hat{i}\left(\underset{\downarrow \atop 0}{\frac{\partial w}{\partial y}} - \underset{\downarrow \atop 0}{\frac{\partial v}{\partial z}}\right) + \hat{j}\left(\underset{\downarrow \atop \gamma}{\frac{\partial u}{\partial z}} - \underset{\downarrow \atop 0}{\frac{\partial w}{\partial x}}\right) + \hat{k}\left(\underset{\downarrow \atop 0}{\frac{\partial v}{\partial x}} - \underset{\downarrow \atop 0}{\frac{\partial u}{\partial y}}\right)$$

$= \gamma \hat{j}$ $\therefore$ only y-comp vorticity is present

(consistent with pinwheel rotating about $\hat{j}$ axis.)

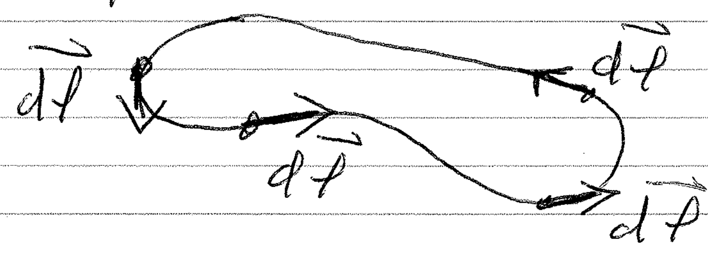
Circulation

Circulation $C \equiv \oint \vec{v} \cdot \vec{dl}$

line integral around a <u>closed curve</u> (also called a <u>contour</u> integral)

directed arc length of curve

$\vec{dl}$ points <u>counterclockwise</u> around curve.



Can think of $\vec{dl}$ as: $\vec{dl} = \hat{l}\, dl$

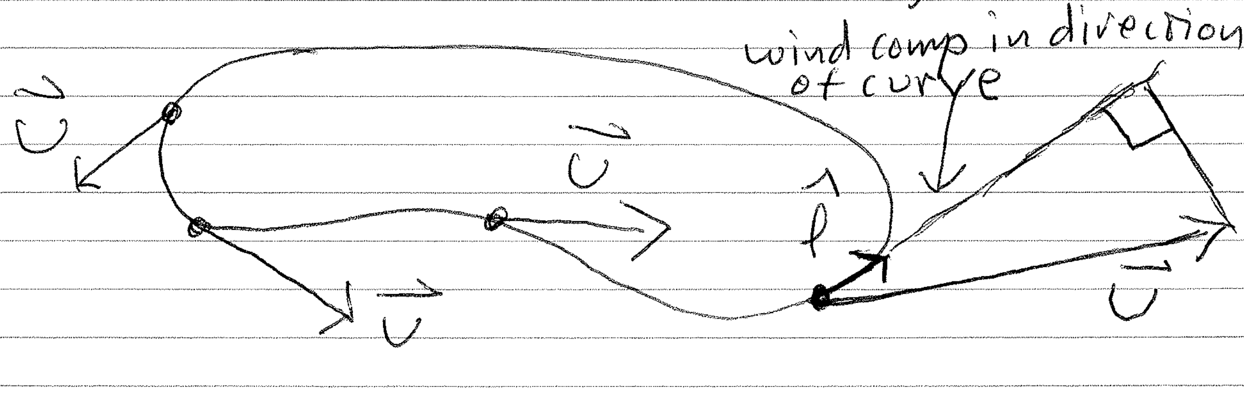
unit vector in direction of curve, pointing counterclockwise

tiny chunk of curve (not a vector)

Can think of $\vec{v} \cdot \vec{dl}$ as:

$$\vec{v} \cdot \vec{dl} = (\vec{v} \cdot \hat{l})\, dl$$

$$= \left(\text{wind component in direction of curve}\right) \text{ times } dl$$



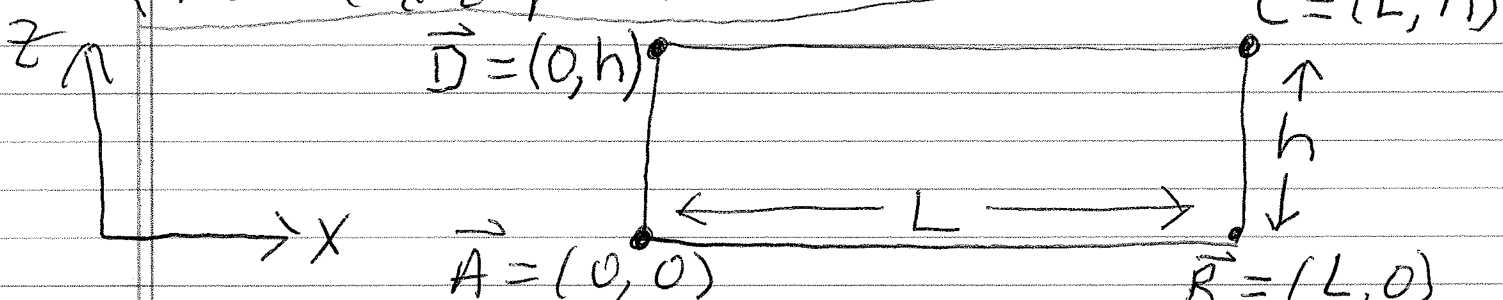
So $C \equiv \oint \vec{v} \cdot d\vec{l} = \oint (\vec{v} \cdot \hat{l})\, dl$

$=$ wind comp in direction of curve <u>integrated</u> <u>over whole curve</u>.

So Average amount of wind tangent to curve is $= \dfrac{C}{\oint dl} = \dfrac{C}{l}$

total length of curve

Let's derive a formula to help us evaluate the circulation around a rectangle of length L and height h in the xz plane.



$$C = \oint \vec{v} \cdot d\vec{l} = \int_{\vec{A} \text{ to } \vec{B}} \vec{v} \cdot d\vec{l} + \int_{\vec{B} \text{ to } \vec{C}} \vec{v} \cdot d\vec{l} + \boxed{\int_{\vec{C} \text{ to } \vec{D}} \vec{v} \cdot d\vec{l}} + \int_{\vec{D} \text{ to } \vec{A}} \vec{v} \cdot d\vec{l}$$

evaluate counter-clockwise

now reverse order on these two

$$\boxed{\vec{v} = u\hat{i} + w\hat{k}}$$

$$= \int_{\vec{A} \text{ to } \vec{B}} \vec{v} \cdot d\vec{l} + \int_{\vec{B} \text{ to } \vec{C}} \vec{v} \cdot d\vec{l} - \int_{\vec{D} \text{ to } \vec{C}} \vec{v} \cdot d\vec{l} - \int_{\vec{A} \text{ to } \vec{D}} \vec{v} \cdot d\vec{l}$$

$$\therefore C = \int_0^L u(x,0)\, dx + \int_0^h w(L,z)\, dz - \int_0^L u(x,h)\, dx - \int_0^h w(0,z)\, dz$$

Using this formula, calculate circulation around this rectangle for case of constant wind: $u = U$ (const)
$v = 0$
$w = 0$

$$\therefore C = \int_0^L U\,dx + 0 - \int_0^L U\,dx - 0$$
$$= UL - UL = 0$$

[recall that for this same wind we found that vorticity was 0]

Now use this formula to calculate circulation around this rectangle when we have unidirectional shear:

$u = \gamma z$, where $\gamma > 0$ is const
$v = 0$
$w = 0$

$$\therefore C = \int_0^L 0\,dx + 0 - \int_0^L \gamma h\,dx - 0$$

$$= -\gamma h L$$

$$= -\gamma A \text{ where } A = hL \text{ is area enclosed by curve}$$

[for this wind profile we saw that there was vort in $\hat{j}$ direction of magnitude γ]