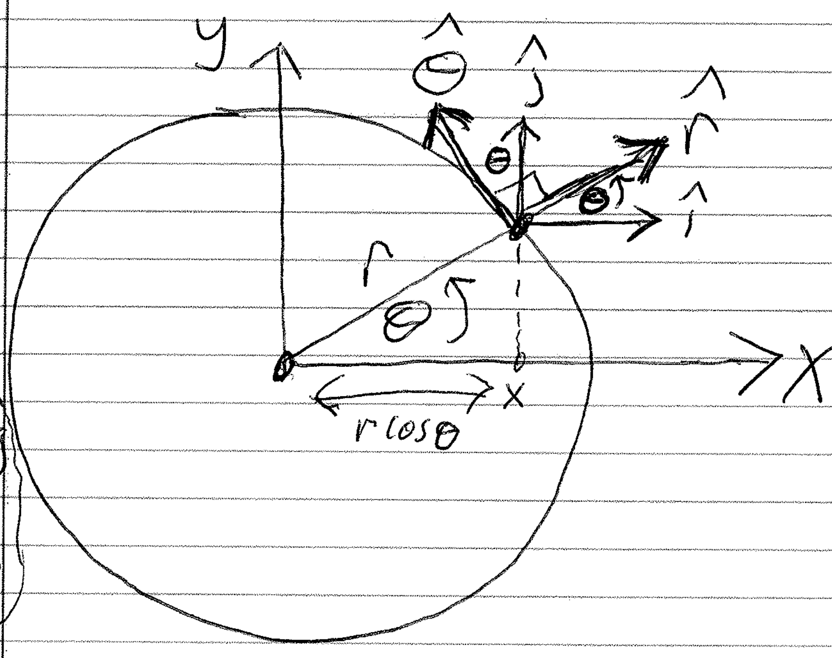
Lecture 11 ①

Kinematics of vorticity and circulation (continued)

Brief review of cylindrical coordinates:



z is pointing out of page

$x = r\cos\theta$
$y = r\sin\theta$

$\therefore x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

(since $\cos^2\theta + \sin^2\theta = 1$)

$\therefore r = \sqrt{x^2+y^2}$

r is radial coordinate (radius)

θ is azimuthal (tangential) coordinate, an angle.

$\hat{r}$ is unit vector in direction of increasing r

$\hat{\theta}$ " " " " " " " θ

We can write the velocity $\vec{u}$ in Cartesian coordinates as:

$$\vec{u} = u\hat{i} + v\hat{j} + w\hat{k}$$

This same velocity can be written in cylindrical coordinates as:

$$\vec{u} = v_r\hat{r} + v_\theta\hat{\theta} + w\hat{k}$$

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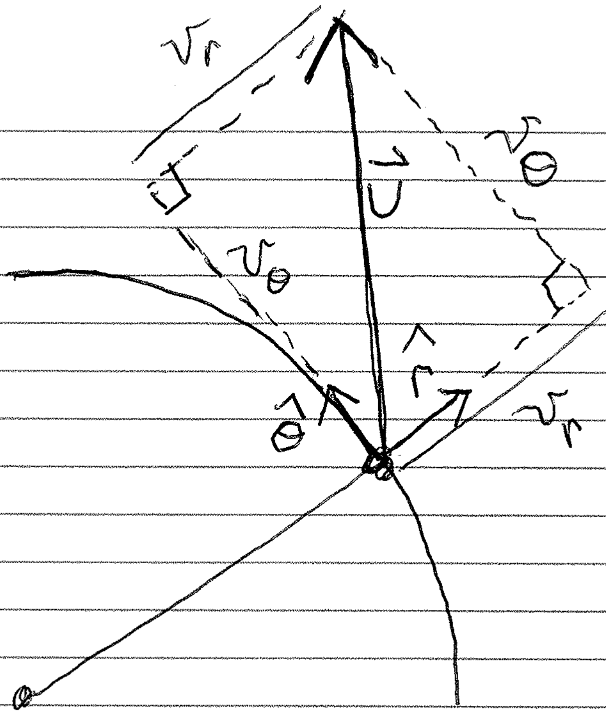


Graphical decomposition of velocity $\vec{U}$ into its radial v_r and azimuthal v_θ components (in this example $w=0$).

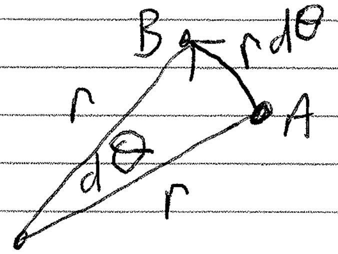
radial velocity v_r means $\frac{dr}{dt}$ rate of change of an air parcel's radial coordinate

azimuthal velocity v_θ means $r\frac{d\theta}{dt}$ it's radius times rate of change of an air parcel's azimuthal coordinate.

angular velocity

why must radius be here? Velocity represents a change of distance over a time interval

That is, radius times angular velocity.



The distance travelled from A to B (chunk of circle) is radius times angle subtended ($d\theta$)

angular velocity: $\frac{d\theta}{dt}$

azimuthal (tangential velocity): $r\frac{d\theta}{dt}$

called v_θ but this is what it means

So can write angular velocity as $\frac{v_\theta}{r}$

... and angular momentum is $v_\theta r$

and vertical vorticity $\zeta \equiv \hat{k} \cdot \vec{\omega}$
$= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

Can be expressed in cylindrical coords as:

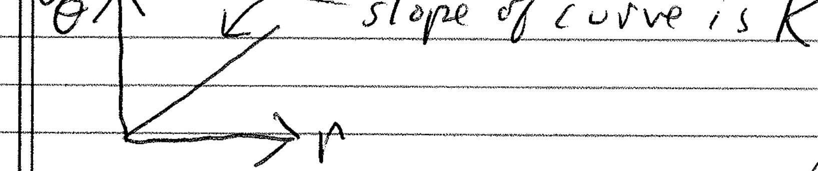
$$\zeta = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

A special kind of vortex:
solid body vortex (also called solid body rotation).

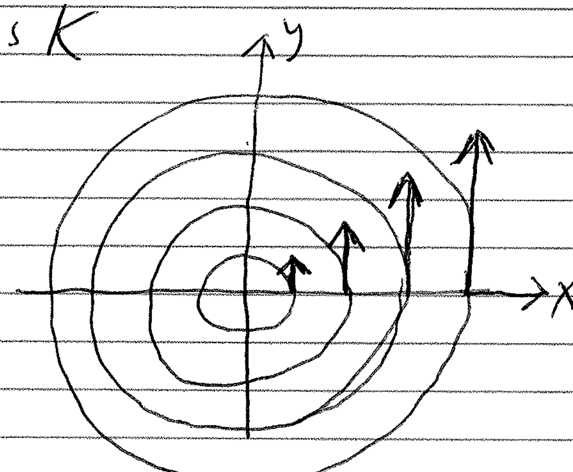
In a solid body vortex: $\boxed{\begin{array}{l} v_\theta = Kr \\ v_r = 0 \\ w = 0 \end{array}}$ $K = \text{const}$

So $\vec{v} = Kr\hat{\theta}$

Since $v_\theta = Kr$, v_θ increases linearly with radius, like a record player (or planet).



Streamlines with superimposed velocity vectors:



Calculate angular velocity for this vortex:

$$\text{ang velocity} = \frac{v_\theta}{r} = \frac{Kr}{r} = K, \text{ a constant!}$$

So ang velocity is the same everywhere. And we've found the physical meaning of "K". It's the angular velocity.

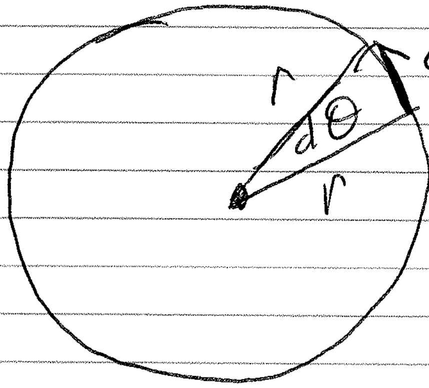
Calculate the vertical vorticity for this vortex:

$$\xi = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = \frac{\partial}{\partial r}Kr + \frac{Kr}{r} - 0$$

$$= K + K = 2K$$

also const!

Suppose we have a solid body vortex. Calculate the circulation around a circle of radius r centered on the vortex.



$$\vec{dl} = |\vec{dl}|\,\hat{\theta} = r\,d\theta\,\hat{\theta}$$

$$\therefore\ C = \oint \vec{v}\cdot\vec{dl} = \int_0^{2\pi} \vec{v}\cdot r\hat{\theta}\,d\theta$$

θ changes on circle but r doesn't change

$$= r\int_0^{2\pi} \vec{v}\cdot\hat{\theta}\,d\theta$$

$$= r\int_0^{2\pi} (v_r\hat{r} + v_\theta\hat{\theta} + w\hat{k})\cdot\hat{\theta}\,d\theta$$

$$= r\int_0^{2\pi} v_\theta\,d\theta$$

$v_\theta = Kr$ in a solid body vortex

$$= r\int_0^{2\pi} Kr\,d\theta$$

$$= Kr^2\int_0^{2\pi} d\theta$$

$$= 2\pi K r^2$$

use $\pi r^2 = A$ area of circle

$$= 2KA$$

[recall that the vorticity in this problem was $2K$]