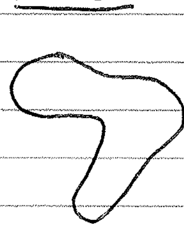
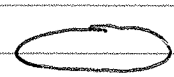
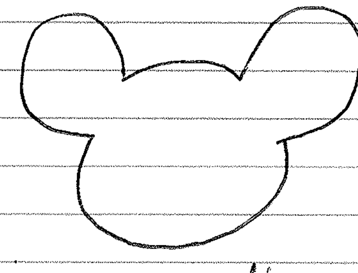
Lecture 12

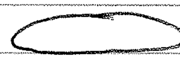
Stokes' Theorem

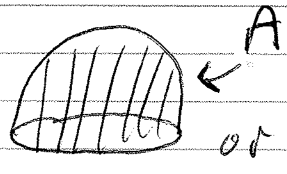
→ Consider any closed curve drawn in the flow,

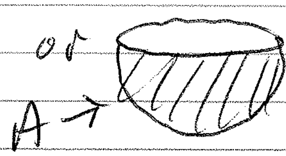
e.g.  or  or 

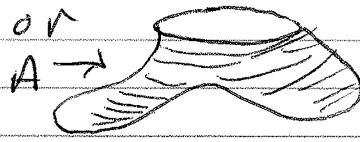
[These can be "flat" curves in two-dimensions or crazy wobbly distorted curves in three-dimensions. It doesn't matter]

→ and also consider any area bounded by that ~~cur~~ curve,

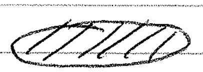
e.g. for the 2-D curve  these are acceptable areas:



or 

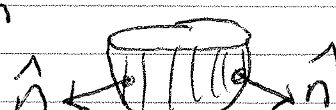
or 

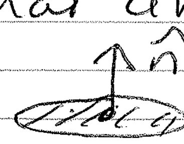
or think of a "jiffy pop®" self-contained stove-top pan area.

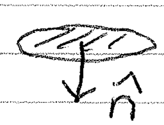
or  ← A in this case the area is two-dimensional just like the curve.

→ and consider the local unit outward normal vector ~~to~~ $\hat{n}$ to that area:

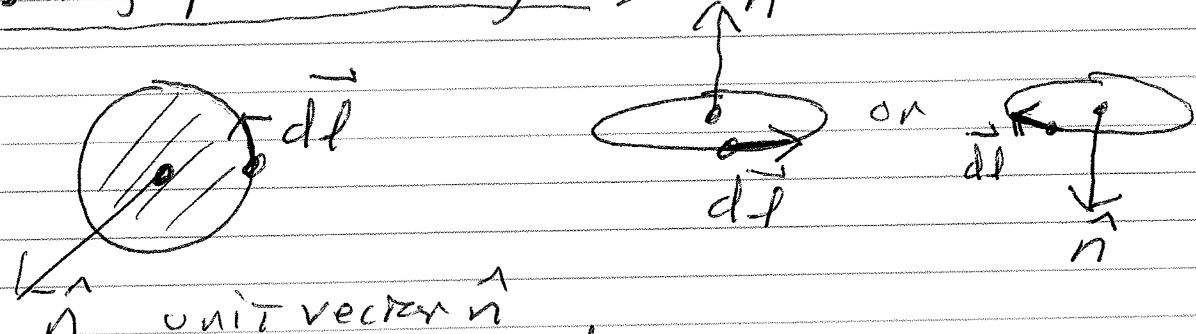
"normal" means perpendicular

or 

or 

or 

→ and consider the tiny directed element of the curve $d\ell$ evaluated counter-clockwise [to an observer whose face is being pierced by $\hat{n}$]



unit vector $\hat{n}$ must be upward (out of page) if $d\ell$ is as indicated

→ and consider any vector field $\vec{v}$ defined on the curve and on the area enclosed by curve.

[We will often consider "$\vec{v}$" to be the velocity vector, but it doesn't have to be. In Stokes' theorem $\vec{v}$ can represent temp gradient or vorticity or acceleration or...]

The only constraint on the "$\vec{v}$ field" is that $\vec{v}$ must be continuous with continuous derivatives except possibly at isolated points.

Whew... okay... now we can state

Stokes Theorem:

$$\boxed{\oint \vec{v}\cdot\vec{d\ell} = \iint \hat{n}\cdot(\nabla\times\vec{v})\,dA}$$

So, if we interpret $\vec{v}$ as velocity then

→ $\oint \vec{v} \cdot d\vec{l}$ is circulation C

→ $\nabla \times \vec{v}$ is vorticity $\vec{\omega}$

→ $\hat{n} \cdot (\nabla \times \vec{v})$ is the vorticity normal to the area bounded by the curve, and Stokes then means:

Circulation = area integrated normal component of vorticity

Symbolically, this means

$$C = \iint \hat{n} \cdot \vec{\omega} \, dA$$ where $\vec{\omega}$ is the vorticity vector

÷ by area A

$$\frac{C}{A} = \frac{1}{A} \iint \hat{n} \cdot \vec{\omega} \, dA$$ → this is an integral divided by interval, so it's an average. It's the average normal component of vorticity. Call it $\overline{\omega_n}$

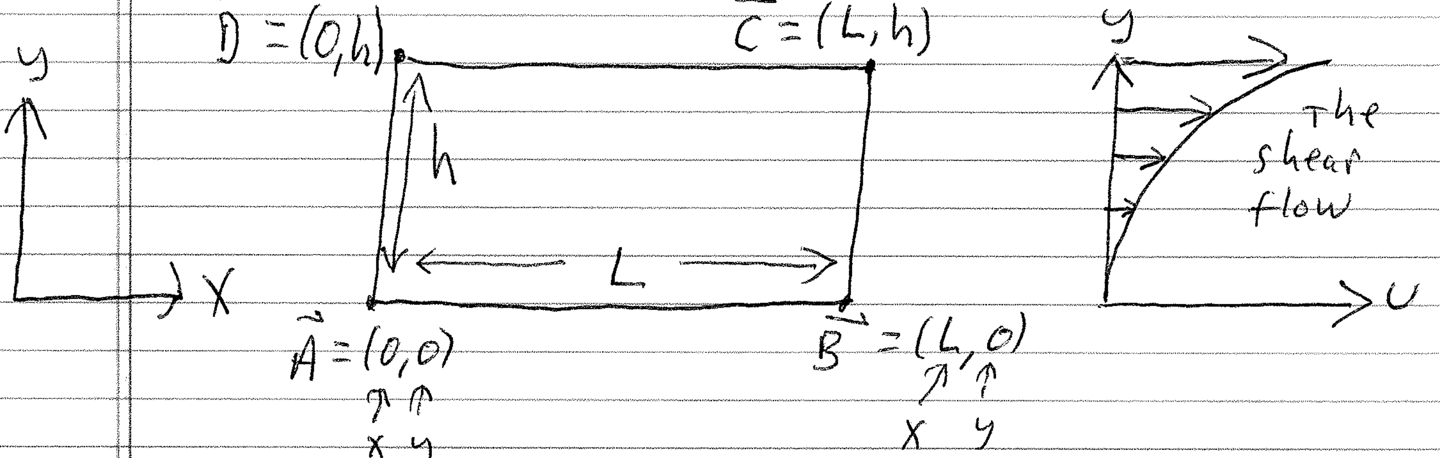
$$\therefore \frac{C}{A} = \overline{\omega_n}$$

So the average normal-component of vorticity is the circulation divided by area.

$\boxed{\text{Exercise}}$ Prove that Stokes' Theorem is true for the case of a horizontal rectangle (considered below) drawn in the nonlinear shear flow given by:

$$u = \sigma y^2 \quad (\text{where } \sigma > 0 \text{ is a constant})$$
$$v = 0$$
$$w = 0$$

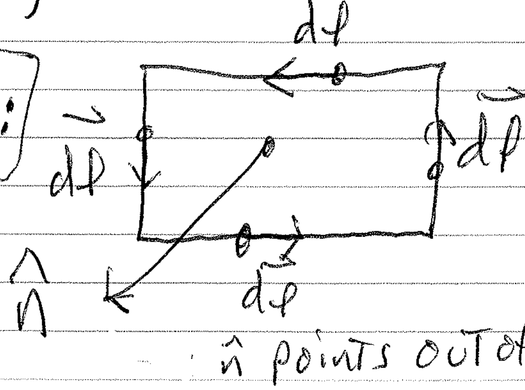
Consider the rectangle's southwest corner to coincide with $x=0, y=0$. Rectangle has lengths L in x-direction and h in y-direction:



Okay, let's prove it.

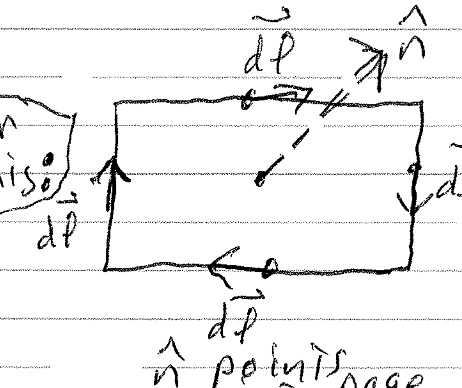
First we should establish direction of $\hat{n}$ and of the $d\vec{\ell}$. Remember the $d\vec{\ell}$ should go around the curve counterclockwise to an observer who's face is pierced by $\hat{n}$.

So we can use this:



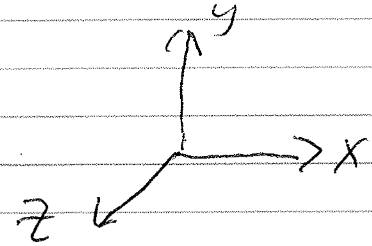
$\hat{n}$ points out of page

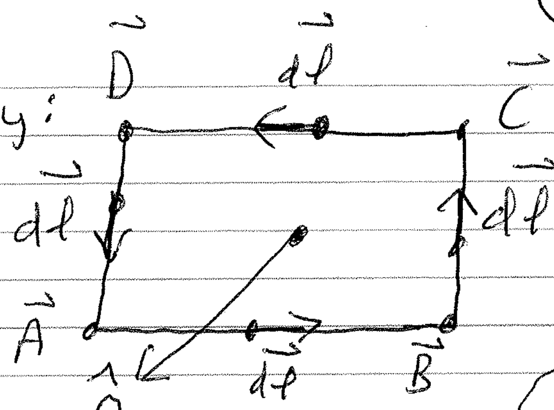
or This:



$\hat{n}$ points into page

I like the first way:





So $\hat{n}$ points in the z direction. So $\boxed{\hat{n} = \hat{k}}$

Let's calculate the circulation:

$$C = \oint \vec{u}\cdot\vec{dl} = \int_{\vec{A}\text{ to }\vec{B}} \vec{u}\cdot\vec{dl} + \int_{\vec{B}\text{ to }\vec{C}} \vec{u}\cdot\vec{dl} + \int_{\vec{C}\text{ to }\vec{D}} \vec{u}\cdot\vec{dl} + \int_{\vec{D}\text{ to }\vec{A}} \vec{u}\cdot\vec{dl}$$

note sign switcheroo on last two integrals $\rightarrow$

$$+\int_{\vec{A}\text{ to }\vec{B}} \vec{u}\cdot\vec{dl} + \int_{\vec{B}\text{ to }\vec{C}} \vec{u}\cdot\vec{dl} - \int_{\vec{D}\text{ to }\vec{C}} \vec{u}\cdot\vec{dl} - \int_{\vec{A}\text{ to }\vec{D}} \vec{u}\cdot\vec{dl}$$

use $v = 0$ $\rightarrow$

$$= \int_0^L u(x,0)\,dx + 0 - \int_0^L u(x,h)\,dx - 0$$

use fact that $u = \sigma y^2$ $\rightarrow$

$$= \int_0^L \sigma(0)^2\,dx - \int_0^L \sigma h^2\,dx = -\sigma h^2 L$$

$$\therefore \boxed{C = -\sigma h^2 L}$$

Now let's calculate the normal comp of vorticity:

$$\hat{n}\cdot(\nabla\times\vec{u}) = \hat{k}\cdot(\nabla\times\vec{u}) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$\hat{n} = \hat{k}$

$$= \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(\sigma y^2)$$

$$= -2\sigma y$$

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Now calculate the area-integrated normal component of vorticity:

$$\iint \hat{n}\cdot(\nabla\times\vec{u})\,dA = \int_0^h \int_0^L (-2\sigma y)\,dx\,dy$$

y integral ↗ ↑ x integral

$$= -2\sigma \int_0^h y[x]\Big|_0^L dy$$

$$= -2\sigma \int_0^h yL\,dy = -2\sigma L \int_0^h y\,dy$$

$$= -2\sigma L\left[\frac{y^2}{2}\right]\Big|_0^h$$

$$= -\frac{\cancel{2}\sigma L h^2}{\cancel{2}} = -\sigma L h^2$$

$$\therefore\ \boxed{\iint \hat{n}\cdot(\nabla\times\vec{u})\,dA = -\sigma L h^2}$$

and also

$$\boxed{C = -\sigma h^2 L}$$

They're the same!

$$\boxed{\therefore\ \oint \vec{u}\cdot d\vec{l} = \iint \hat{n}\cdot(\nabla\times\vec{u})\,dA}$$

So we've proved that Stokes Th^{m} holds up in this case.