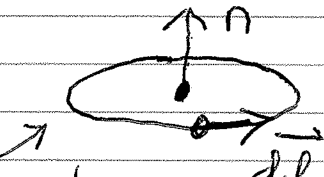
Lecture 13 ①

Kinematics of Vorticity and Circulation (continued)

Apply Stokes' Thm to a succession of tiny (infinitesimal) closed plane curves centered on a point of interest, with each curve smaller than the last,



Tiny plane curve enclosing tiny area A

Consider these curves in the limit of $A \to 0$ as they close in on that point of interest,

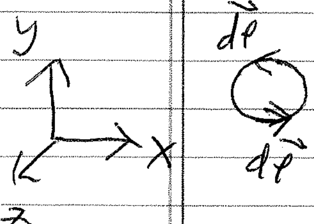
$$\therefore \lim_{A\to 0} \oint \vec{U}\cdot\vec{d\ell} = \lim_{A\to 0} \iint \hat{n}\cdot(\nabla\times\vec{U})\,dA$$

$$= \lim_{A\to 0} \big[\underbrace{\hat{n}\cdot(\nabla\times\vec{U})}_{\text{at point of interest}}\big] A$$

$$(\star)\quad \boxed{\hat{n}\cdot(\nabla\times\vec{U}) = \lim_{A\to 0} \frac{C}{A}}$$

So vertical vort component can be thought of, using $\star$ with $\hat{n} = k$, as

$$\zeta = \hat{k}\cdot(\nabla\times\vec{U}) = \lim_{A\to 0}\frac{C}{A}$$

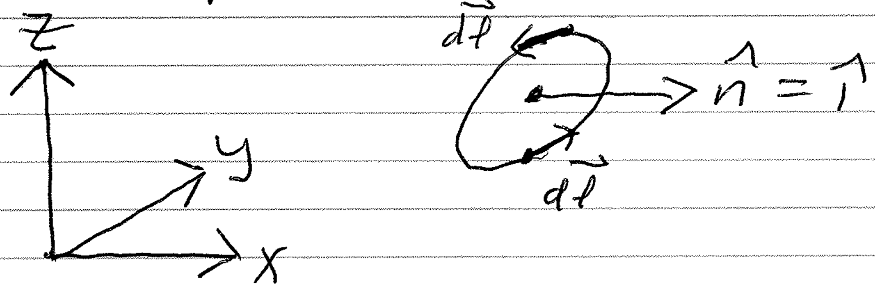


where the curve must be drawn in the <u>horizontal</u> plane [in order for $\hat{n} = \hat{k}$]

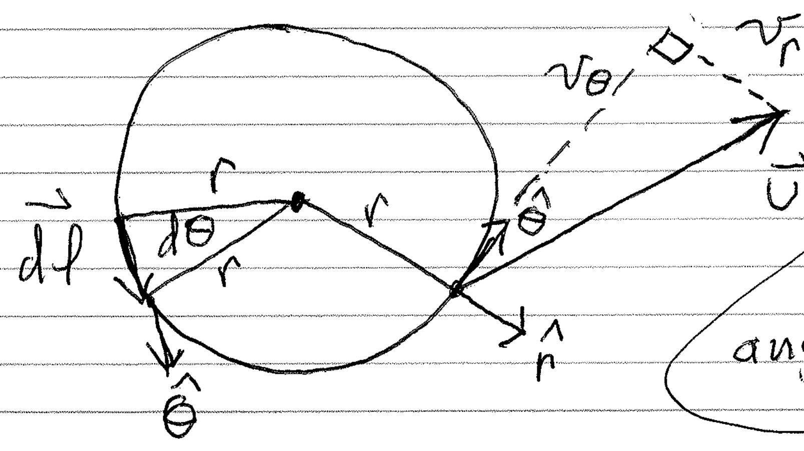
The x-comp of vorticity can be thought of, using (✡) as

$$\hat{\imath}\cdot(\nabla\times\vec{v}) = \lim_{A\to 0}\frac{C}{A}$$

where the <u>normal to the curve</u> must be $\hat{\imath}$ and that means the curve must be drawn in the yz plane (so $\hat{n}=\hat{\imath}$):



Eq^n (✡) is valid for any plane curve but we get a particularly nice result if we consider a <u>circle</u>:



recall:

$v_\theta = \hat{\theta}\cdot\vec{v}$

$\vec{d\ell} = r\,d\theta\,\hat{\theta}$

angular velocity is $\frac{v_\theta}{r}$

From (✡): applied to circle curve

$$\hat{n}\cdot(\nabla\times\vec{v}) = \lim_{A\to 0}\frac{C}{A} = \lim_{A\to 0}\frac{\oint \vec{v}\cdot\vec{d\ell}}{A}$$

$$= \lim_{r\to 0}\frac{\oint \vec{v}\cdot r\,d\theta\,\hat{\theta}}{\pi r^2}$$

[having $A\to 0$ means $r\to 0$]

$$= \lim_{r \to 0} \frac{1}{\pi r} \int_0^{2\pi} \vec{v} \cdot \hat{\theta}\, d\theta = \lim_{r \to 0} \frac{1}{\pi r} \int_0^{2\pi} v_\theta \, d\theta$$

$$= \lim_{r \to 0} \frac{1}{\pi} \int_0^{2\pi} \underbrace{\left(\frac{v_\theta}{r}\right)}_{\text{angular velocity}} d\theta$$

$$= \lim_{r \to 0} 2\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{v_\theta}{r}\, d\theta\right)$$

Want to use ave = $\frac{\text{integral}}{\text{interval}}$. In this case interval is 2π

$$= 2\bar{\Omega}$$

where $\bar{\Omega} \equiv \lim_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} \frac{v_\theta}{r}\, d\theta$

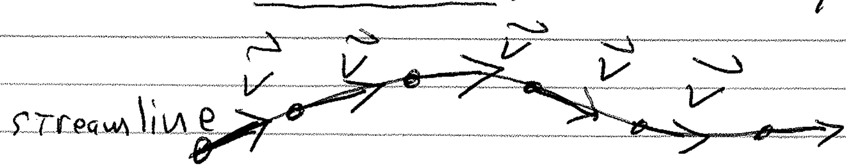
$\bar{\Omega}$ is the <u>local</u> <u>angular velocity</u>.

So <u>Vorticity is Twice Local angular velocity</u>

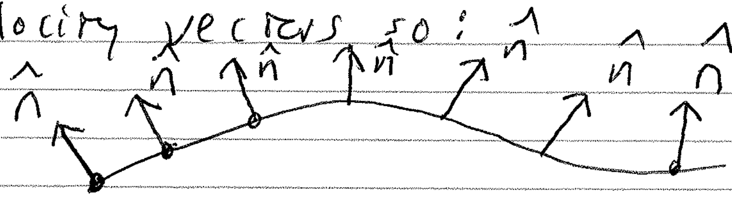
Brief review of natural coordinates. (in 2 dimensions)

Consider natural coords defined with respect to <u>streamlines</u> (instead of trajectories)

Here's a streamline. It's everywhere tangent to velocity vectors

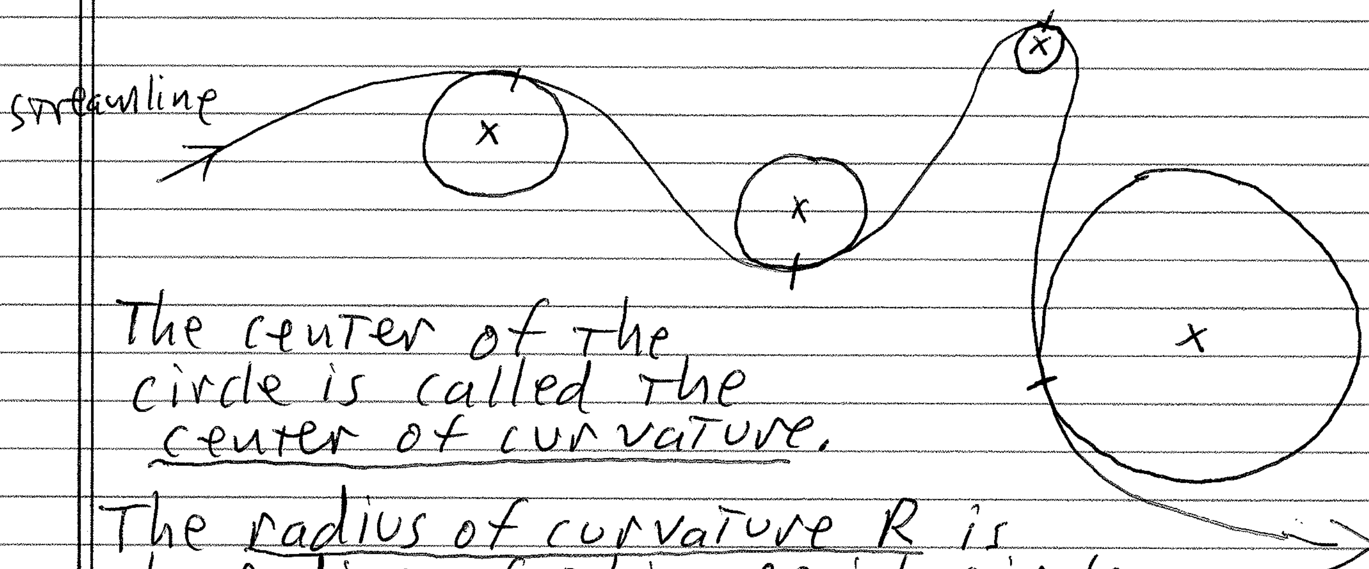


Let $\hat{n}$ be unit vector $\perp$ and to <u>left</u> of the velocity vectors so:



<u>Note: This $\hat{n}$ has nothing to do with $\hat{n}$ in Stokes' Th^m! They're completely unrelated.</u>

Now, for every point on streamline consider a circle that touches that point with <u>same tangent</u> and <u>curvature</u> as streamline at that point.



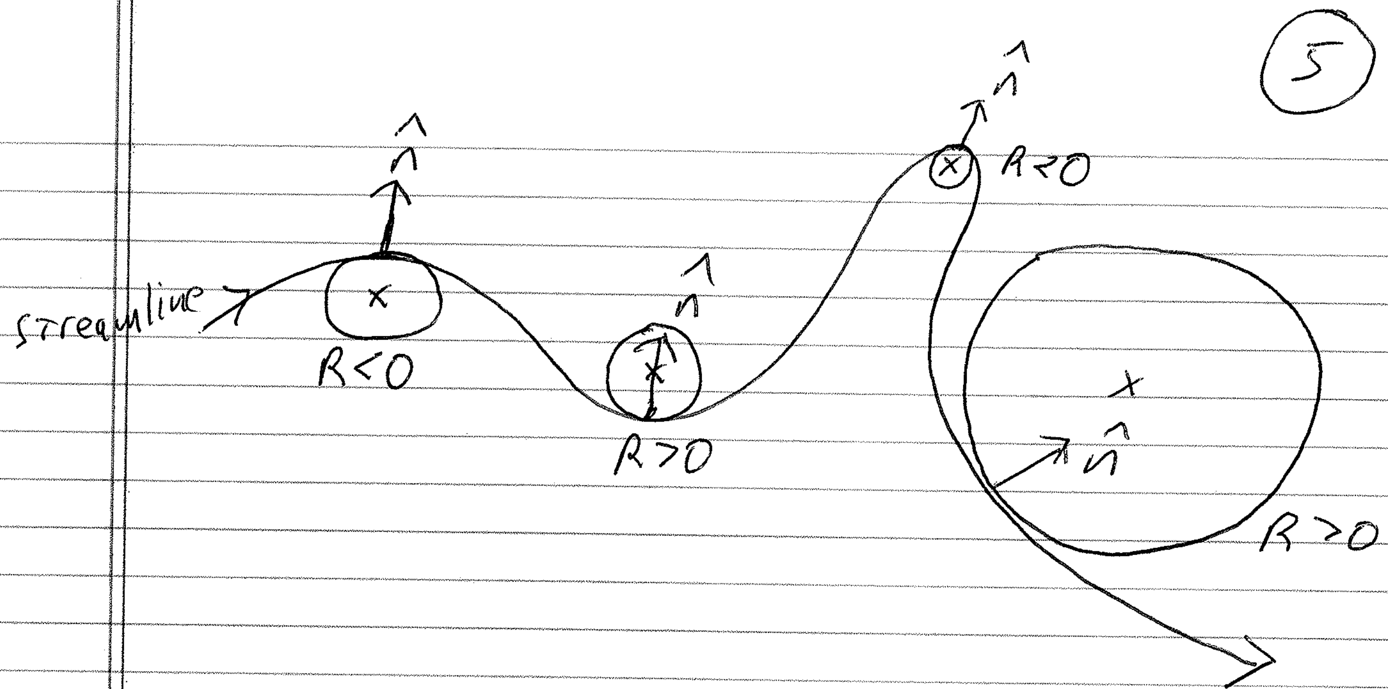
The center of the circle is called the <u>center of curvature</u>.

The <u>radius of curvature R</u> is the radius of this special circle.

Note:

- R varies from point to point along streamline.
- R is <u>small</u> (small circle) where <u>curvature is large</u>.
- Flatter curves have bigger R. Straight line has <u>infinite R</u>.

<u>Sign convention</u>: R is <u>positive</u> where $\hat{n}$ points <u>toward</u> center of curvature and <u>negative</u> where $\hat{n}$ points <u>away</u> from center of curvature.



So if streamline curves toward left then $R > 0$.
" " " " right " $R < 0$.

End of review

Vertical vorticity is related to flow in the xy plane (u, v winds)

x-comp vorticity is related to flow in the yz plane (v, w winds)

y-comp vorticity is related to flow in the xz plane (u, w winds).

Can work with natural coordinates in any of those planes.

Vertical vorticity in natural coordinates:

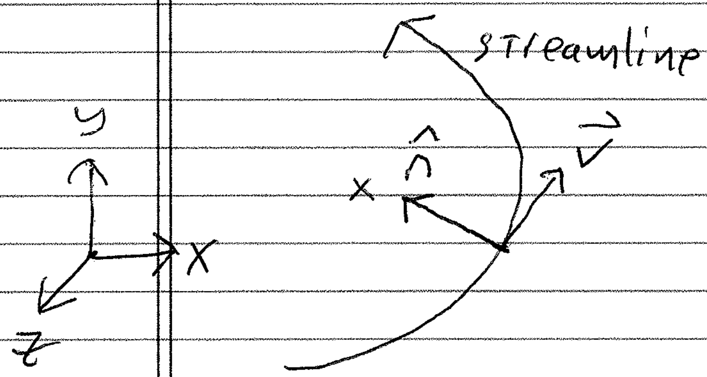
$$\boxed{\zeta = -\frac{\partial V}{\partial n} + \frac{V}{R}}$$

where V is horizontal wind speed. Flow is considered in the xy plane.

$-\frac{\partial V}{\partial n}$ is <u>shear vorticity</u>

$\frac{V}{R}$ is <u>curvature vorticity</u>

e.g. <u>Curvature vorticity in a cyclonic vortex</u>
(in N. hemisphere)



Since $\hat{n}$ is directed toward center of curvature, $R > 0$,

And since speed V is always positive,

$\frac{V}{R} > 0$ positive curvature vort.

[out of page curvature vort]

→ <u>Right hand rule for curvature vorticity:</u>

align (curl) fingers of right hand along streamline, pointing in direction of flow. Thumb indicates direction of curvature vorticity.