Lecture 14

Kinematics of Vorticity and Circulation (continued)

Vertical vorticity in <u>natural coordinates</u>: <u>[based on streamlines]</u>

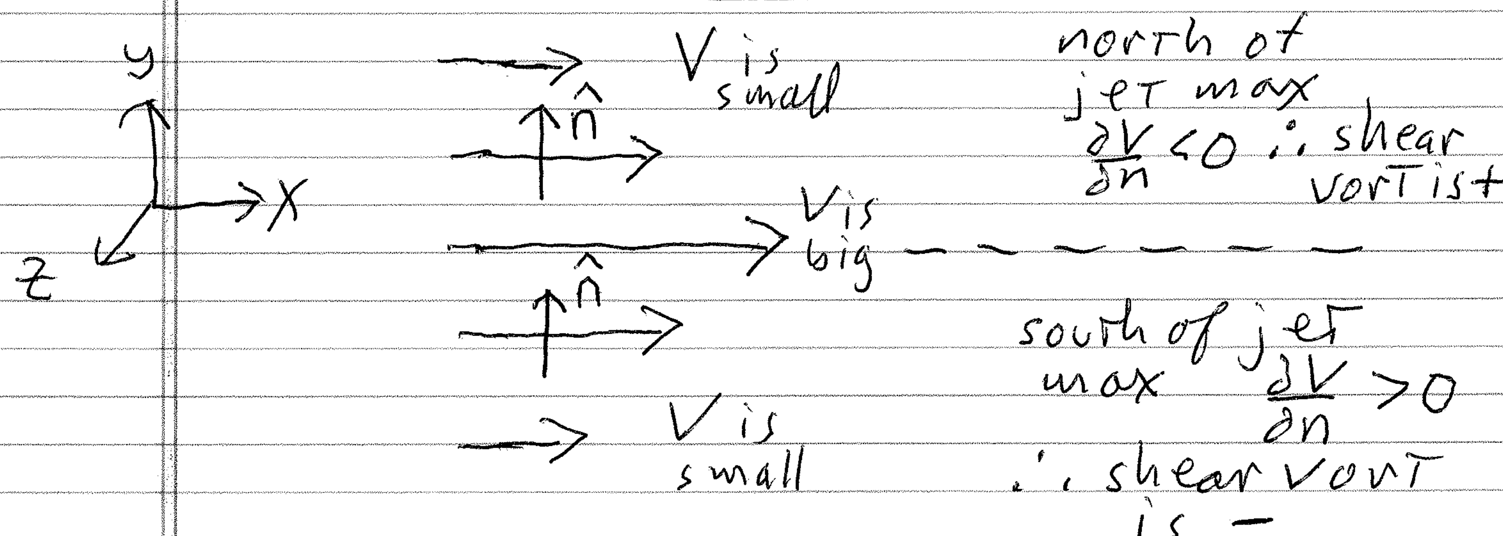
$$\zeta = -\frac{\partial V}{\partial n} + \frac{V}{R}$$

$\boxed{\frac{\partial V}{\partial n} \text{ is } \hat{n}\cdot\nabla V}$ — shear vorticity, curvature vorticity

where V is the wind speed $V=\sqrt{u^2+v^2}$ associated with <u>horizontal</u> flow.

Analogous equations hold for the other two vorticity components (and speeds based on x, z and y, z velocity components).

e.g. <u>shear vorticity in jet stream</u>



Shear vorticity where V is largest (at jet max) is <u>zero</u>!

→ Right hand rule for shear vorticity: align the fingers of your right hand with velocity vector, then curl fingers towards slower flow (smaller V). Thumb indicates the direction of vorticity.

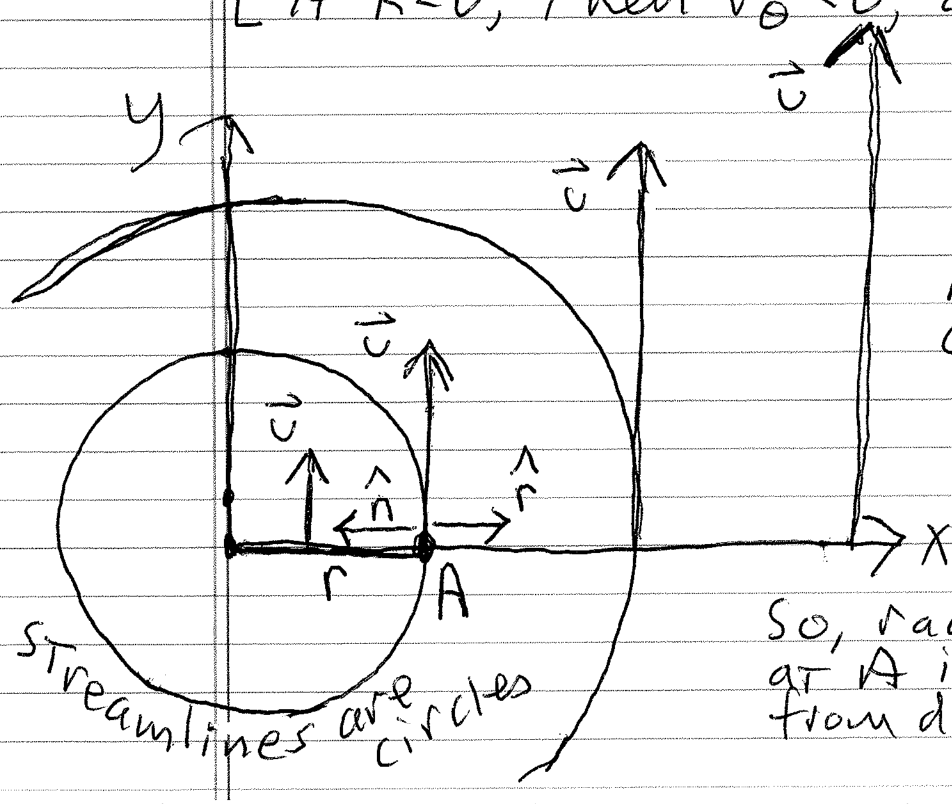
Apply right hand rule for shear vort to the jet stream example. Quickly get: out-of-page shear vort north of jet max (so pos vert vort) and into page shear vort south of jet (so neg vert vort).

→ Revisit the solid body vortex, $v_\theta = Kr$. [using cylindrical coords we already found that $\zeta = 2K$ for that vortex].

$v_\theta = Kr$

Let $K > 0$. So $v_\theta > 0$ everywhere. So $\boxed{V = v_\theta}$

[If $K < 0$, then $v_\theta < 0$ and then $V = -v_\theta$]



Go to any location (say point A in diagram).

At A, $\hat{n}$ points toward center of vortex, which is also the center of curvature for the streamline passing through A.

So, radius of curvature R at A is positive. And we see from diagram that $R = r$.

From right hand rule for curvature vort, what is sense of curvature vort at A?

From right hand rule for shear vort, what is sense of shear vort at A?

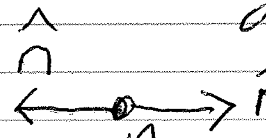
Now let's explicitly calculate the shear and curvature vorticities in this example:

Curvature vort:

$$\frac{V}{R} = \frac{v_\theta}{r} = \frac{Kr}{r} = K \quad (>0)$$

Shear vort:

$$-\frac{\partial V}{\partial n} = -\hat{n}\cdot\nabla V = -(-\hat{r})\cdot\nabla V$$



$$= \hat{r}\cdot\nabla V = \frac{\partial V}{\partial r}$$

$$= \frac{\partial v_\theta}{\partial r} = \frac{\partial}{\partial r}(Kr)$$

$$= K \quad (>0)$$

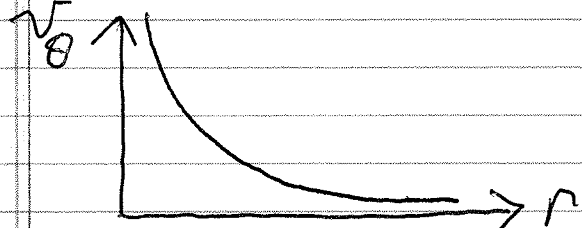
So, shear vort and curvature vort are the same (both $= K$)

$$\therefore\ \zeta = K + K = 2K \quad \text{(as before)}.$$

It should be clear from diagram that there was nothing special about point A. Get same result at any location in solid body vortex.

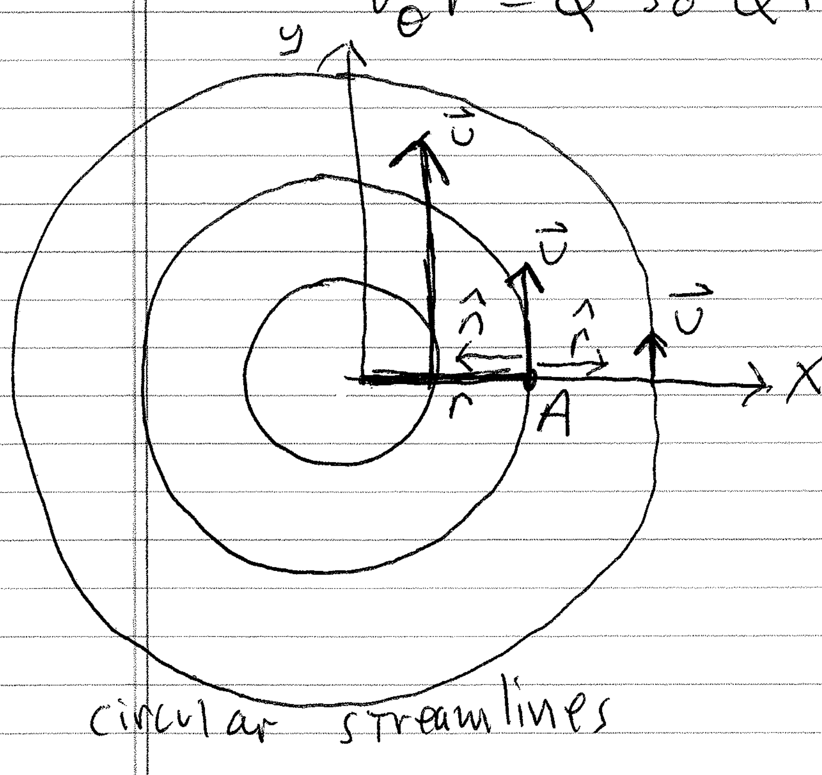
→ **Another example** Consider vorticity in a "vr - vortex" (also known as a line vortex, potential vortex or irrotational vortex).

Azimuthal velocity in a vr-vortex: $\boxed{v_\theta = \frac{Q}{r}}$ where Q is a constant. (Let $Q > 0$)



v_θ is small where r is big.
v_θ is big where r is small.
Note that v_θ blows up (becomes infinite, is "singular" at $r=0$).

In this vortex the angular momentum ($v_\theta r$) is constant (indep of radius).
$v_\theta r = Q$ so Q is the (const) ang momentum.



What do the two right hand rules tell you about the shear vort and curvature vort in this example?

circular streamlines

Now let's explicitly calculate the shear and curvature vorticities.

As in prev example we have $V = v_\theta$ (since $v_\theta > 0$), $\hat{n} = -\hat{r}$ and $R = r$. So:

Curvature vort:

$$\frac{V}{R} = \frac{v_\theta}{r} = \frac{Q}{r^2} \quad (>0)$$

Shear vort:

$$-\frac{\partial V}{\partial n} = -\hat{n} \cdot \nabla V = -(-\hat{r}) \cdot \nabla V = \hat{r} \cdot \nabla V$$

$$= \frac{\partial V}{\partial r} = \frac{\partial v_\theta}{\partial r} = \frac{\partial}{\partial r}\frac{Q}{r} = -\frac{Q}{r^2} \quad (<0)$$

So the shear and curvature vorticities are equal and opposite!

$$\zeta = \underbrace{-\frac{Q}{r^2}}_{\substack{\uparrow \\ \text{shear} \\ \text{vort}}} + \underbrace{\frac{Q}{r^2}}_{\substack{\uparrow \\ \text{curvature} \\ \text{vort}}} = 0 \; !$$

... well not so fast. $\zeta = 0$ everywhere except at $r = 0$ where $\frac{Q}{r^2}$ becomes undefined (division by 0).

So what happens at $r = 0$? Using Stokes theorem, you can show that the vorticity ζ is infinite at $r = 0$.