Lecture 15

Kinematics of Vorticity and Circulation (continued)

Relative versus absolute vorticity.

To an observer in a rotating reference frame [e.g. an observer on rotating earth], the velocity of an air parcel will be different from the air parcel velocity observed by an observer in a non-accelerating or "fixed" ref frame.

Non-accelerating ref frame ↔ inertial ref frame

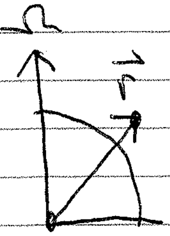
Accelerating ref frame ↔ non-inertial ref frame

A rotating ref frame is an example of an accelerating ref frame.

Recall (from Dynamics I) that:

$$\boxed{\vec{u}_a = \vec{u} + \vec{\Omega} \times \vec{r}}$$

$\vec{u}_a$	$\vec{u}$	$\vec{\Omega} \times \vec{r}$
absolute velocity [velocity observed in an inertial reference frame]	relative velocity [velocity obs in rotating ref frame.]	velocity of the rotating ref frame at location $\vec{r}$



where $\vec{\Omega}$ is angular velocity of earth

and $\vec{r}$ is position vector of an air parcel measured with respect to the center of earth.

So absolute vorticity is:

$$\vec{\omega}_a \equiv \nabla \times \vec{U}_a = \nabla \times (\vec{U} + \vec{\Omega} \times \vec{r})$$

$$= \underbrace{\nabla \times \vec{U}}_{\text{relative vorticity}} + \underbrace{\nabla \times (\vec{\Omega} \times \vec{r})}_{\text{earth vorticity}}$$

$$\therefore \vec{\omega}_a = \vec{\omega} + \underbrace{\nabla \times (\vec{\Omega} \times \vec{r})}_{\text{Can show it's just } 2\vec{\Omega}!}$$

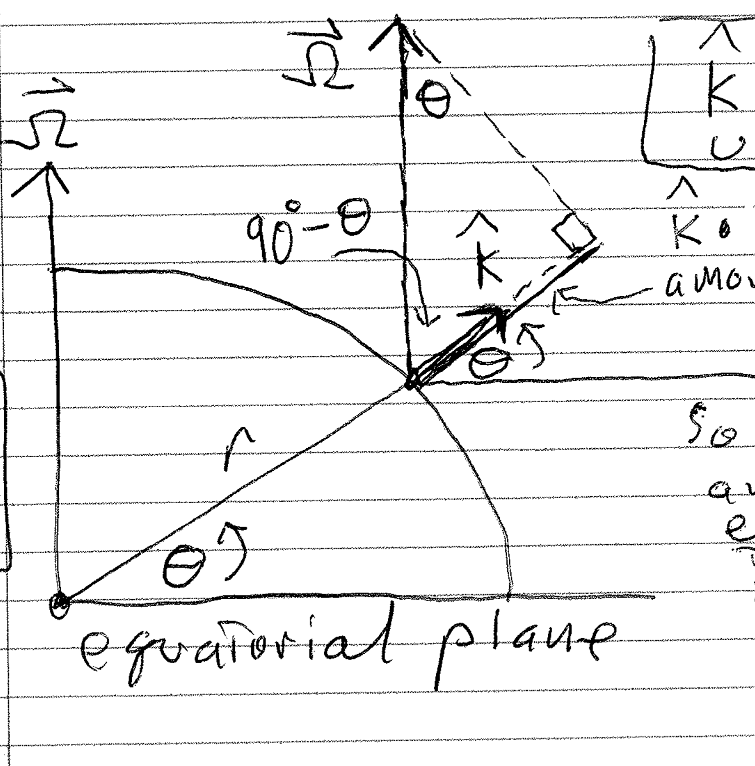
[done in prob. set]

$$\therefore \boxed{\vec{\omega}_a = \vec{\omega} + 2\vec{\Omega}}$$

absolute vort = relative vort + earth vort [Twice angular velocity of earth]

→ How much earth vorticity is in the local vertical direction?

$\hat{k}$ is "local up" unit vector.



θ is latitude

$\hat{k} \cdot \vec{\Omega}$ is amount of $\vec{\Omega}$ in $\hat{k}$ direction

So $\hat{k} \cdot 2\vec{\Omega}$ is amount of earth vorticity in $\hat{k}$ direction.

From diagram we see that:

$$\sin\theta = \frac{\hat{k}\cdot\vec{\Omega}}{|\vec{\Omega}|} = \frac{\hat{k}\cdot\vec{\Omega}}{\Omega}$$ where $\Omega \equiv |\vec{\Omega}|$

$\therefore$ $\hat{k}\cdot\vec{\Omega} = \Omega\sin\theta$ mult by 2

$$\hat{k}\cdot 2\vec{\Omega} = \underbrace{2\Omega\sin\theta}_{\text{This is } f\text{, the Coriolis parameter.}}$$

So $f \equiv 2\Omega\sin\theta$ is the component of the earth vorticity about local vertical axis.

Absolute versus relative vertical vorticity:

vector vorticity: $\vec{\omega}_a = \vec{\omega} + 2\vec{\Omega}$ take $\hat{k}\cdot$ where $\hat{k}$ is local vertical unit vector

$$\underbrace{\hat{k}\cdot\vec{\omega}_a}_{\text{call it } \eta} = \underbrace{\hat{k}\cdot\vec{\omega}}_{\text{call it } \zeta} + \underbrace{\hat{k}\cdot 2\vec{\Omega}}_{\text{it's } f}$$

$$\eta = \zeta + f$$

absolute vertical vorticity — relative vertical vorticity — vertical comp of earth vorticity

Vorticity Dynamics

So far we've looked at kinematics of vorticity and circulation: their definitions, physically what they mean, how they're related, etc. Now we'll study how they evolve in time: vorticity dynamics.

Want a prognostic equation for ξ. Note that:

$$\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)$$

So, to get an equation for $\frac{\partial \xi}{\partial t}$ (i.e. showing how various physical forcings contribute to $\partial \xi / \partial t$, take the x-derivative of the y-component equation of motion and subtract from it the y-derivative of the x-component equation of motion.

x : east
y : north
z : up

$\boxed{\text{x-comp eq}^{n} \text{ of motion}}$

viscosity coefficient

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \nu \nabla^2 u$$

or :

Expand $\frac{Du}{Dt}$

$$(1) \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \nu \nabla^2 u$$

y-comp eq^n of motion

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \nu\nabla^2 v$$

or:

$$(2)\quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \nu\nabla^2 v$$

Take $\frac{\partial}{\partial x}(2)$ minus $\frac{\partial}{\partial y}(1)$:

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}\left(w\frac{\partial v}{\partial z}\right)$$

$$- \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(w\frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{1}{\rho}\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial x}(fu) - \frac{\partial}{\partial y}(fv)$$

$$+ \frac{\partial}{\partial x}\left(\nu\nabla^2 v\right) - \frac{\partial}{\partial y}\left(\nu\nabla^2 u\right)$$

Examine left hand side of this eq^n, one pair of terms at a time:

<u>Local derivative terms.</u>

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial \zeta}{\partial t}$$

Terms originating from advection by u-wind
(i.e. $u\frac{\partial}{\partial x}$ terms):

$$\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right)-\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}\right)$$

interchange order of differentiation

$$=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial}{\partial x}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-u\frac{\partial}{\partial y}\frac{\partial u}{\partial x}$$

$$=\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$=\zeta\frac{\partial u}{\partial x}+u\frac{\partial \zeta}{\partial x}$$