Lecture 16 ①

Derivation of the vertical vorticity eqⁿ (cont'd)

Now look at the terms that originated from advection by the v-comp wind, i.e. $\boxed{v\frac{\partial}{\partial y}}$ terms

$$\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right)-\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)$$

interchange order of differentiation

$$=\boxed{\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}+v\frac{\partial}{\partial x}\frac{\partial v}{\partial y}\boxed{-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}}-v\frac{\partial}{\partial y}\frac{\partial u}{\partial y}$$

$$=\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$=\zeta\frac{\partial v}{\partial y}+v\frac{\partial \zeta}{\partial y}$$

Now look at terms originating from advection by w-comp velocity, the $\boxed{w\partial/\partial z}$ terms:

$$\frac{\partial}{\partial x}\left(w\frac{\partial v}{\partial z}\right)-\frac{\partial}{\partial y}\left(w\frac{\partial u}{\partial z}\right)$$

interchange order of differentiation

$$=\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}+w\frac{\partial}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}-w\boxed{\frac{\partial}{\partial y}\frac{\partial u}{\partial z}}$$

$$=\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}+w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$=\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}+w\frac{\partial \zeta}{\partial z}$$

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Now work with righthand side of $\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$ equation:

Pressure terms

$$\frac{\partial}{\partial x}\left(-\frac{1}{\rho}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$

[I cancelled 2 minuses in last term]

$$= -\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial y} \boxed{-\frac{1}{\rho}\frac{\partial}{\partial x}\frac{\partial p}{\partial y}} + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial x} \boxed{+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)}$$

They cancel since $\frac{\partial}{\partial x}\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\frac{\partial p}{\partial x}$ [just interchange order of differentiation]

Scratch Paper

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) = \frac{\partial}{\partial x}\rho^{-1} = -\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}$$

same deal for $\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)$

$$= \frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}$$

Coriolis terms

$$-\frac{\partial}{\partial x}(fu) - \frac{\partial}{\partial y}(fv)$$

$$= -f\frac{\partial u}{\partial x} - f\frac{\partial v}{\partial y} - v\frac{df}{dy}$$

$$= -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{df}{dy}v$$

since f varies only with latitude, it varies with y (the north/south coordinate), not x or z!

$\boxed{f = f(y)}$

<u>friction Terms</u>

$$\frac{\partial}{\partial x}\left(\nu \nabla^2 v\right) - \frac{\partial}{\partial y}\left(\nu \nabla^2 u\right)$$

$$= \nu \nabla^2 \frac{\partial v}{\partial x} - \nu \nabla^2 \frac{\partial u}{\partial y}$$

$$= \nu \nabla^2 \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= \nu \nabla^2 \zeta$$

again, interchange order of differentiation $\frac{\partial}{\partial x}\nabla^2 = \nabla^2 \frac{\partial}{\partial x}$

Put 'em all together, get:

$$\frac{\partial \zeta}{\partial t} \boxed{+ \zeta \frac{\partial u}{\partial x}} + u\frac{\partial \zeta}{\partial x} \boxed{+ \zeta \frac{\partial v}{\partial y}} + v\frac{\partial \zeta}{\partial y}$$

$$+ \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + w\frac{\partial \zeta}{\partial z}$$

put these together, get $\zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$

$$= \frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x} - \frac{df}{dy}v$$

$$- f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \nu \nabla^2 \zeta$$

Isolate $\frac{\partial \zeta}{\partial t}$ (put all other terms over onto right hand side).

Get vertical vorticity eq^n:

$$\underbrace{\frac{\partial \zeta}{\partial t}}_{\text{local derivative}} = \underbrace{-u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y} - w\frac{\partial \zeta}{\partial z}}_{\text{advection terms}} + \underbrace{\frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z}}_{\text{tilting terms}}$$

$$\underbrace{+\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}}_{\text{baroclinic terms}} \underbrace{-\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(\zeta + f)}_{\text{stretching term}}$$

$$\underbrace{-v\frac{df}{dy}}_{\text{earth vorticity advection}} + \underbrace{\nu\nabla^2\zeta}_{\text{diffusion of vorticity}}$$

Let's tidy this equation up a little:

$$u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} = \vec{u}\cdot\nabla\zeta$$

$\delta \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is <u>horizontal divergence</u>

$$\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x} = \hat{k}\cdot\left(\frac{1}{\rho^2}\nabla\rho \times \nabla p\right)$$

Vertical Vorticity Eqn

$$\boxed{\begin{aligned}\frac{\partial \zeta}{\partial t} &= -\vec{u}\cdot\nabla\zeta + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} \\ &\quad + \hat{k}\cdot\left(\frac{1}{\rho^2}\nabla\rho \times \nabla p\right) - \delta(\zeta + f) \\ &\quad - v\frac{df}{dy} + \nu\nabla^2\zeta\end{aligned}}$$

Look at each term separately:

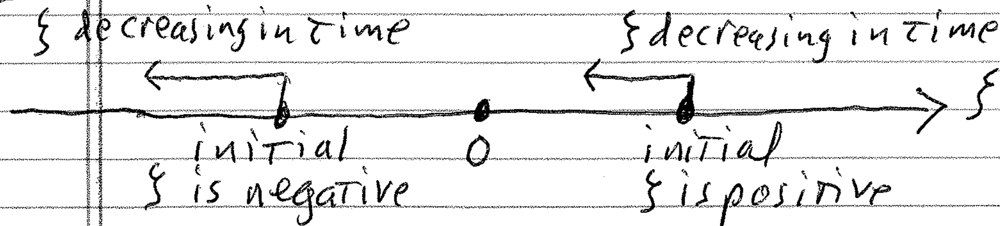
Local deriv term

$\frac{\partial \zeta}{\partial t}$ is rate of change of relative vertical vorticity ζ at a fixed point.

It's forced by contributions from all the terms on the right hand side.

Dif btw value and magnitude

If $\frac{\partial \zeta}{\partial t} < 0$ then $\zeta \downarrow$ with time. In this case, if ζ is positive then ζ decreases in magnitude, but if ζ is negative then it increases in magnitude.

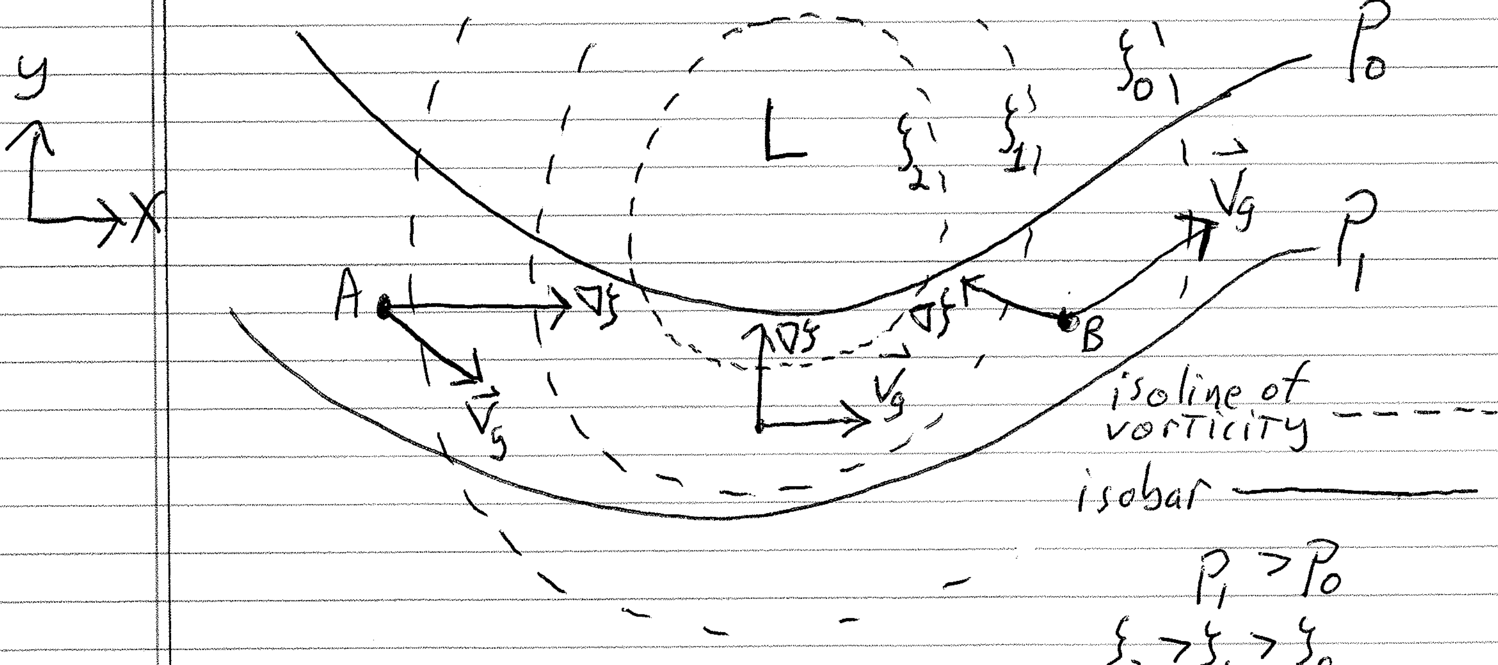


Similarly, if $\frac{\partial \zeta}{\partial t} > 0$ then $\zeta \uparrow$ with time. In this case, if ζ is positive then ζ increases in magnitude but if ζ is negative then ζ decreases in magnitude.

Advection Term

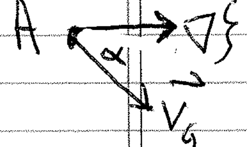
$$\frac{\partial \zeta}{\partial t} = \dots \underbrace{\boxed{-\vec{U} \cdot \nabla \zeta}}_{\text{advection term}}$$

e.g., consider winds and vorticity around a mid-latitude low pressure system at a mid-level of atmosphere, so $\vec{v}$ is close to $\vec{V}_g$.



The vorticity pattern is typical of mid-latitude lows. ξ is often highest near low p since curvature of isobars is largest there (geostrophic flow streamlines have large positive curvature there — small R — so large positive curvature vort there).

Southwest of low:

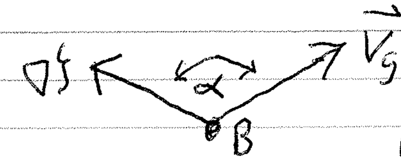


$\boxed{\alpha < 90^\circ}$

$\therefore \; -\vec{v}\cdot\nabla\xi = -|\vec{v}||\nabla\xi|\cos\alpha < 0$
$\quad\;\; - \quad + \quad + \quad +$

$\therefore$ negative vorticity advection (NVA). Tends to make $\partial\xi/\partial t$ negative, i.e. $\xi\downarrow$

Southeast of low:



$\boxed{\alpha > 90^\circ}$

$\therefore \; -\vec{v}\cdot\nabla\xi = -|\vec{v}||\nabla\xi|\cos\alpha > 0$
$\quad\;\; - \quad + \quad + \quad -$

$\therefore$ positive vorticity advection (PVA). Tends to make $\partial\xi/\partial t$ ~~negative~~ positive. So $\xi\uparrow$