Lecture 19

Earth vorticity advection term

$$\frac{\partial \zeta}{\partial t} = \ldots \boxed{-v\frac{df}{dy}}$$

Since f only changes in the north/south direction, $f = f(\phi)$ or $f = f(y)$ where y points north.

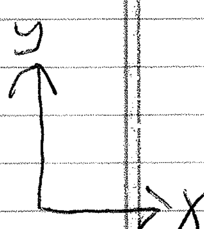
latitude

Advection of earth vorticity by wind is:

$$-\vec{v}\cdot\nabla f = -u\underbrace{\frac{\partial f}{\partial x}}_{0} - v\frac{\partial f}{\partial y} - w\underbrace{\frac{\partial f}{\partial z}}_{0} = -v\frac{\partial f}{\partial y} \left[= -v\frac{df}{dy}\right]$$

So, only the y-component of the wind (v) can advect earth vorticity.

Consider two different wind scenarios (in the N. hemisphere)



westerly wind

$\vec{v} = u\hat{i}$ where $u > 0$

$\longrightarrow \vec{v}$

Since $v = 0$, $-v\frac{df}{dy} = 0$

$\therefore \frac{\partial \zeta}{\partial t} = 0$. No affect on vertical vorticity generation.

northerly wind

f big (high lat)

$\vec{v} = v\hat{j}$ where $v < 0$

f small (low lat)

$-v\frac{df}{dy} = - - + > 0$

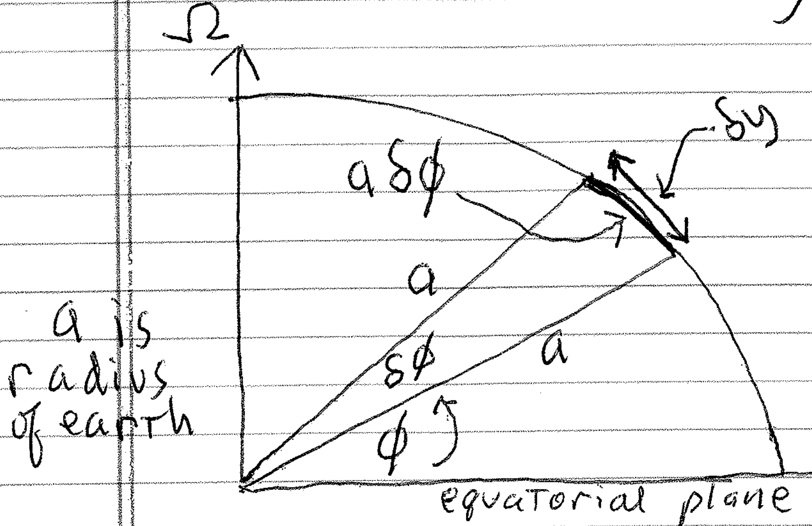
$\therefore \frac{\partial \zeta}{\partial t} > 0$. Positive earth vorticity advection by northerly wind.

Since f increases with latitude (in northern hemisphere), df/dy is positive (in northern hemisphere). But what exactly is df/dy? Let's evaluate it:

$$\frac{df}{dy} = \frac{d}{dy}(2\Omega \sin\phi) = 2\Omega \frac{d}{dy} \sin\phi$$

$$= 2\Omega \left[\frac{d}{d\phi}(\sin\phi)\right]\frac{d\phi}{dy} \quad \text{[chain rule!]}$$

$$= 2\Omega \cos\phi \, \frac{d\phi}{dy} \quad \text{so what is } \frac{d\phi}{dy}\,?$$



The straight line distance δy becomes equal to the circular arc distance $a\,\delta\phi$ in limit of $\delta\phi \to 0$ ($\delta y \to 0$):

$\boxed{\delta y = a\,\delta\phi}$ for $\delta\phi \to 0$ [$\delta y \to 0$]

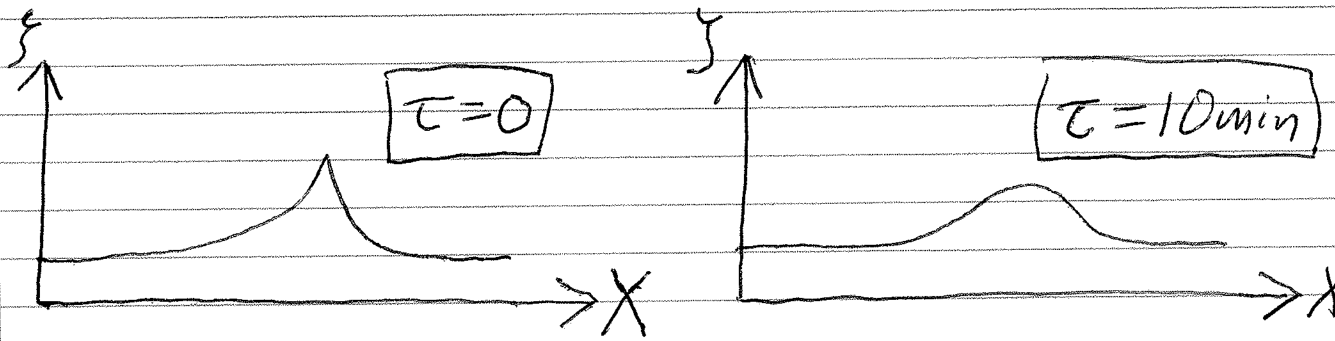
$$\therefore \frac{d\phi}{dy} = \lim_{\delta y \to 0} \frac{\delta\phi}{\delta y} = \frac{\delta\phi}{a\,\delta\phi} = \frac{1}{a}$$

$$\therefore \frac{df}{dy} = \frac{2\Omega \cos\phi}{a}$$

Diffusion of vorticity

$$\frac{\partial \zeta}{\partial \tau} = \ldots \boxed{+\nu \nabla^2 \zeta}$$

Molecular diffusion acts to smooth out large gradients of ζ. Peaks get less peaky.

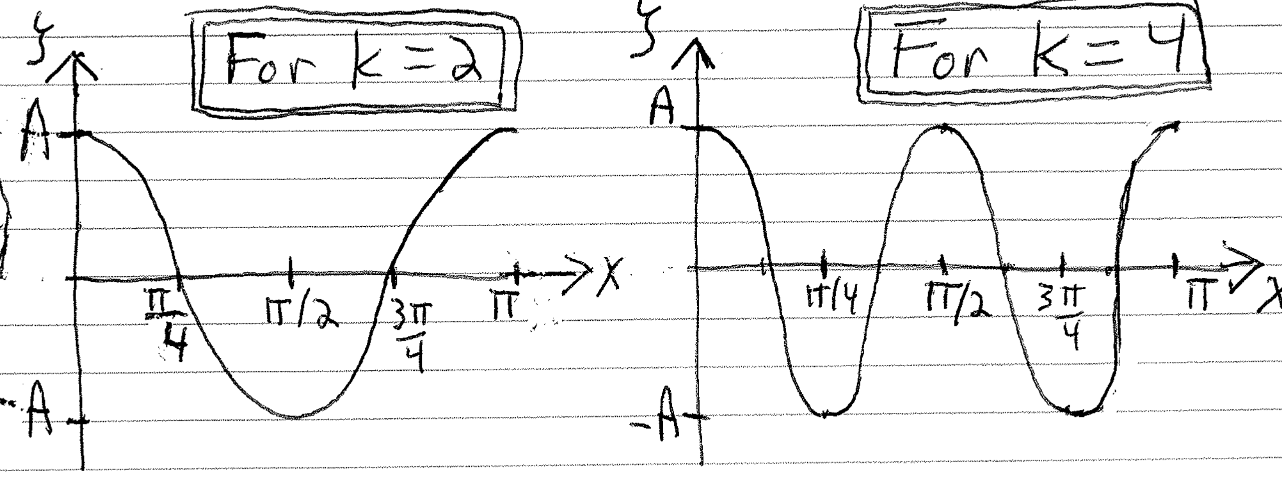


→ e.g. Suppose we have a vorticity distribution at $\tau=0$ given by:

$$\zeta(x,0) = A\cos kx,$$

where A and k are constants. k is called the wavenumber because the bigger k is, the more waves there are (in a given distance). Bigger k also means bigger gradient of ζ.

initial ζ distributions for 2 different k



See how the vorticity changes with time with diffusion as the only forcing term:

$$\frac{\partial \zeta}{\partial t} = \nu \nabla^2 \zeta$$

Since the initial condition is a function only of x, we anticipate that ζ will remain a function ~~only~~ of x (but not of y or z). $\therefore$ $\nabla^2 \zeta$ reduces to $\frac{\partial^2 \zeta}{\partial x^2}$.

$$\therefore \quad (\star) \quad \frac{\partial \zeta}{\partial t} = \nu \frac{\partial^2 \zeta}{\partial x^2}$$

Seek a trial solution of the form

$$(\star\star) \quad \zeta(x,t) = a(t) \cos kx,$$

where $a(t)$ is an unknown time-dependence. This distribution looks like the initial condition. ["Trial" means try it and see if it works].

Pre-calculate derivatives:

$$\frac{\partial \zeta}{\partial t} = \frac{da}{dt} \cos kx$$

$$\frac{\partial \zeta}{\partial x} = -a\boxed{k} \sin kx$$ [This confirms that bigger k means bigger spatial derivative (gradient)]

$$\frac{\partial^2 \zeta}{\partial x^2} = -a k^2 \cos kx$$

Substitute these into $(\star)$, get:

$$\frac{da}{d\tau}\cos kX = -\nu a k^2 \cos kX \quad \div \text{ by } \cos kX$$

$$\frac{da}{d\tau} = -\nu a k^2$$ 1^{st} order linear homogeneous o.d.e. for $a(\tau)$.

Separate variables and integrate

$$\frac{da}{a} = -\nu k^2 d\tau$$

$$\int_{a(0)}^{a(\tau)} \frac{da'}{a'} = -\int_0^{\tau} \nu k^2 d\tau'$$

$$\left[\ln a'\right]\Big|_{a(0)}^{a(\tau)} = -\nu k^2 \left[\tau'\right]\Big|_0^{\tau}$$

$$\ln a(\tau) - \ln a(0) = -\nu k^2 \tau$$

$$\ln \frac{a(\tau)}{a(0)} = -\nu k^2 \tau$$ exponentiate both sides

$$\frac{a(\tau)}{a(0)} = \exp(-\nu k^2 \tau)$$ mult by $a(0)$

$$a(\tau) = a(0)\exp(-\nu k^2 \tau)$$

To determine $a(0)$, use the initial condition $\xi(X,0) = A\cos kX$ in the trial solution (☆☆). Set $\tau = 0$ in (☆☆):

$$\underbrace{\xi(X,0)}_{A\cos kX} = a(0)\cos kX$$

Use initial condition: $A\cos kX$ $\quad \therefore \boxed{a(0) = A}$

$$\therefore \boxed{a(\tau) = A\exp(-\nu k^2 \tau)}$$

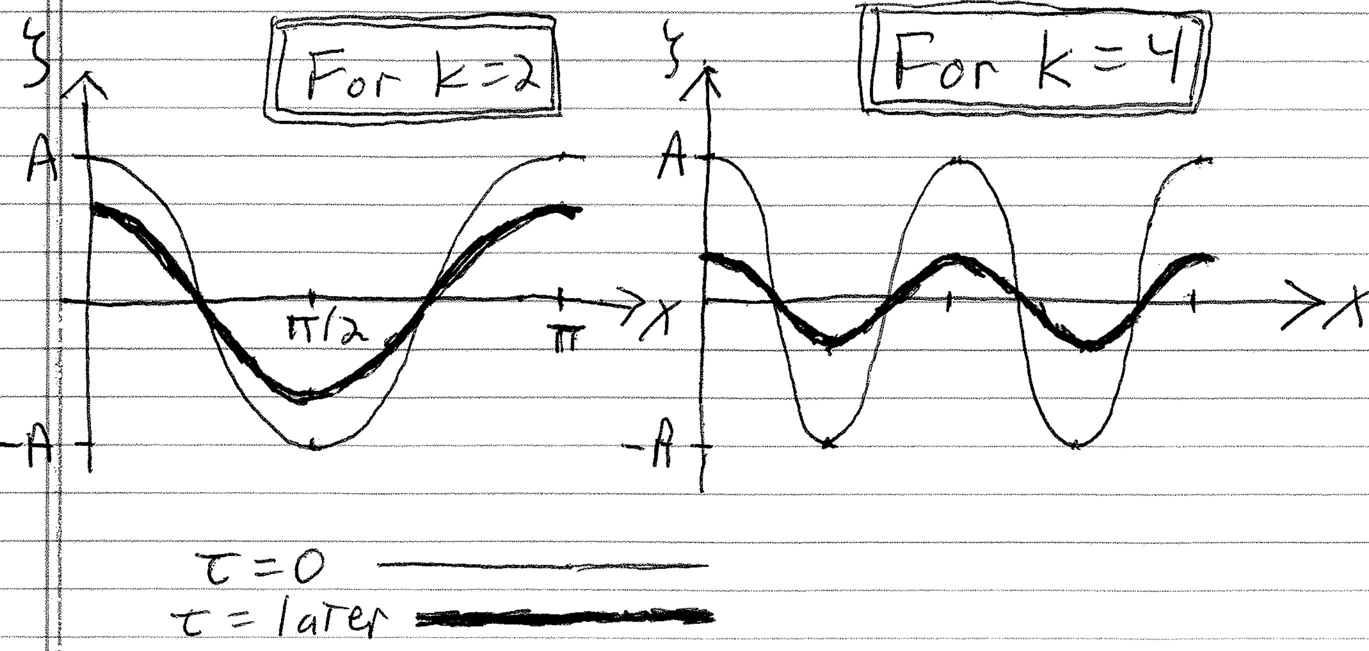
$\therefore$ (☆☆) becomes:

$$(☆☆☆) \quad \boxed{\zeta(x,\tau) = A \exp(-\nu k^2 \tau) \cos kx}$$

It's a pattern that decays with time.

The bigger the νk^2, the faster the decay.

So, fast decay with big viscosity ν and/or large gradients of vorticity (big k).



The $k=4$ profile (bigger gradient) decays faster than the $k=2$ profile.

Let's derive the "e-folding" time scale T_e, which is the time it takes ζ to decay to e^{-1} (about one third) of its initial amplitude.

So T_e is defined by: $\boxed{\zeta(x, T_e) = e^{-1}\zeta(x,0)}$

Plug in the solution ($\cancel{\star\star\star}$) for $\xi(x,\tau)$:

$$A \exp(-\nu k^2 T_e) \cos kx = e^{-1} A \cos kx$$

$\div$ by $A \cos kx$ $\qquad \exp(-\nu k^2 T_e) = e^{-1}$

Take ln of both sides $\qquad -\nu k^2 T_e = -1$

$$\therefore \boxed{T_e = \frac{1}{\nu k^2}}$$

So, faster decay (small T_e) with <u>larger</u> k (smaller sized disturbances).