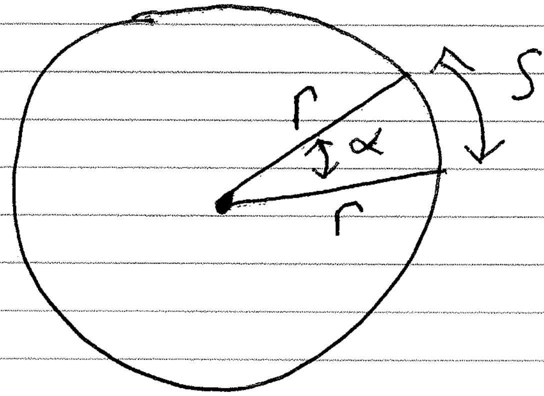
Lecture 2

Review of some pre-requisite math (continued).

7. Arc length of a circle.



If r is the radius of the circle then the arc length S is given by:

$$S = r\alpha$$

S ← arc length; r ← radius; α ← angle subtended

8. Taylor series (Taylor expansion of a function)

For a function f of a single independent variable x, we can express f (in general) as an infinite series of polynomials:

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0}(x-x_0) + \frac{1}{2!}\left.\frac{d^2f}{dx^2}\right|_{x=x_0}(x-x_0)^2$$

$$+ \frac{1}{3!}\left.\frac{d^3f}{dx^3}\right|_{x=x_0}(x-x_0)^3 + \ldots$$

Take the derivatives and then evaluate them at $x=x_0$

where x_0 is an arbitrary expansion point for the series.

Similar-looking formulas appear if the independent variable is y, or z or t.

There's also the multi-dimensional version of Taylor series (not shown).

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9. $\boxed{\text{Taylor approximation}}$

For x "near" the expansion point x_0, $x - x_0$ is small (as are $(x-x_0)^2$ and $(x-x_0)^3$ and ...) and we can approximate $f(x)$ by just retaining the first two terms in the original Taylor series:

$$f(x) \approx f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$$

For x near x_0!

The closer x gets to x_0, the better the approx becomes.

In the limit $x \to x_0$, this "approximation" becomes exact (100% true).

Let's look at some important examples of Taylor approximations.

Example: Taylor approx of $\cos x$ for x small (near 0).

In this case you'll want to choose the expansion point 0, that is: $x_0 = 0$.

So, with $f(x) = \cos x$ we see that $f(x_0) = \cos(x_0) = 1$

x_0 is 0

$$\therefore \quad \frac{df}{dx} = -\sin x$$

$$\therefore \quad \left.\frac{df}{dx}\right|_{\substack{x=x_0 \\ \text{(which is 0)}}} = -\sin x\Big|_{x=0} = 0$$

So the Taylor approx formula $f(x) \approx f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0)$

yields: $\cos x \approx 1$ for x near 0!

Another example Taylor approx of $\sin x$ for x small (near 0). Again, you'll want to choose the expansion point $x_0 = 0$.

$f(x) = \sin x \qquad f(x_0) = \sin \underset{0}{x_0} = 0$

$$\frac{df}{dx} = \cos x$$

$$\left.\frac{df}{dx}\right|_{x=x_0} = \cos \underset{0}{x_0} = 1$$

So the Taylor approx becomes: $\sin x \approx x$

And one more example

Taylor approximation of $\cos x$ for x near $\frac{\pi}{2}$.

So the expansion point is $x_0 = \pi/2$

$$f(x) = \cos x$$

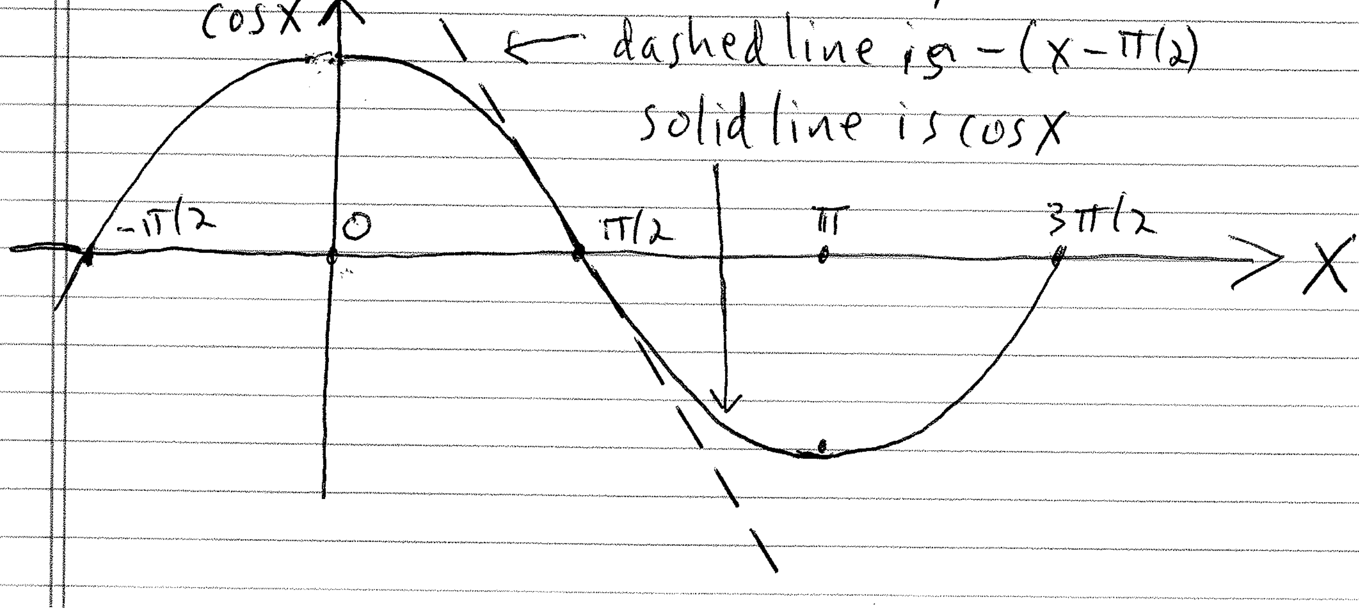
$$f(x_0) = \cos(x_0) = \cos\frac{\pi}{2} = 0$$

$$\frac{df}{dx} = -\sin x$$

$$\left.\frac{df}{dx}\right|_{x=x_0} = -\sin x_0 = -\sin\frac{\pi}{2} = -1$$

$$\therefore \quad \cos x \approx -\left(x - \frac{\pi}{2}\right) \quad \text{for } x \text{ near } \pi/2.$$

Graphically, we can see that the Taylor approx of $\cos x$ in this example yields a good approximation of $\cos x$ near $\pi/2$ but gets pretty bad as x gets away from $\pi/2$:



Important! When working with angles in these examples you must make sure the angles are in <u>radians</u> <u>NOT</u> <u>degrees</u>!!

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10. Multidimensional Taylor approx (3D)

$$f(x,y,z) \approx f(x_0,y_0,z_0) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}}(x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}}(y-y_0) + \left.\frac{\partial f}{\partial z}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}}(z-z_0)$$

valid for x near x_0, y near y_0 and z near z_0

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11. 4D Taylor approx

$$f(x,y,z,\tau) \approx f(x_0,y_0,z_0,\tau_0) + \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) + \frac{\partial f}{\partial z}(z-z_0) + \frac{\partial f}{\partial \tau}(\tau-\tau_0)$$

where all derivatives are evaluated at the same expansion point in space and time, $x=x_0$, $y=y_0$, $z=z_0$, $\tau=\tau_0$

II. Complex numbers

Define the imaginary number i to be the square-root of -1:

$$i \equiv \sqrt{-1}$$

So $i^2 = -1$.

If a and b are two real numbers then the combination,

$c \equiv a + ib$ is a complex number

a is real
b is real
ib is imaginary $c = a + ib$ is complex

So: $\mathrm{Re}(c) = a$ [Real part of c]
$\mathrm{Im}(c) = b$ [Imaginary part of c]

$$\begin{aligned} c^2 &= (a+ib)(a+ib) \\ &= a^2 + 2aib + \overset{\to -1}{i^2} b^2 \\ &= a^2 - b^2 + i2ab \end{aligned}$$

So $\mathrm{Re}(c^2) = a^2 - b^2$
$\mathrm{Im}(c^2) = 2ab$

What are the real and imaginary parts of i^2? $i^2 = -1$ so $\mathrm{Re}(i^2) = -1$
There is no imaginary part.

What are the real and imaginary parts of i^9 ?

$$i^9 = i^8 i^1 = i^2 \, i^2 \, i^2 \, i^2 \, i$$

$$= (-1)(-1)(-1)(-1)\, i$$

$$= + i = 1 i$$

So there is no real part of it, but

$$\mathrm{Im}(i^9) = 1$$

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12. <u>Complex exponential</u>

Euler's formula for the complex exponential:

$$e^{i\phi} = \cos\phi + i \sin\phi$$