Lecture 21 ①

Scale analysis of vertical vorticity eq^n for mid-latitude synoptic-scale flows (continued)

Typical mid-lat $f \sim 10^{-4} s^{-1}$

" " " $\frac{df}{dy} = \frac{2\Omega\cos\phi}{a} \sim \frac{10^{-4} s^{-1} \times 1}{6\times10^{6} m}$

$\therefore \boxed{\frac{df}{dy}} \sim \frac{0.2\times10^{-4} s^{-1}}{10^{6} m} = 0.2\times10^{-10} s^{-1} m^{-1}$ $\quad \boxed{\frac{1}{6} = \frac{2}{12} \approx \frac{2}{10} = .2}$

$= 2\times10^{-11} s^{-1} m^{-1} \approx 10^{-11} s^{-1} m^{-1}$

call it β

Digression on order of magnitude math

reg math	order of mag math
$1+1=2$	$1+1 \approx 1$
$1+1+1=3$	$1+1+1 \approx 1$
$1-1=0$	$1-1 =$ don't know because these are probably not exactly 1

So $1-1 =$ small but how small? $10^{-1}, 10^{-5}, 10^{-19}$?

Typical magnitude of ξ:

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \sim \frac{U}{L} = \frac{10\,m/s}{10^6 m} = 10^{-5} s^{-1}$$

So $\xi << f$! Relative vorticity is usually an order of magnitude smaller than earth's vorticity for typical mid-latitude synoptic-scale flows.

local deriv:

$$\frac{\partial \xi}{\partial t} \sim \max\left(\frac{U/L}{T_{horiz}}, \frac{U/L}{T_{vert}}\right)$$

$$= \max\left(\frac{10^{-5} s^{-1}}{10^5 s}, \frac{10^{-5} s^{-1}}{10^6 s}\right)$$

$$= \max\left(10^{-10} s^{-2}, 10^{-11} s^{-2}\right) = \boxed{10^{-10} s^{-2}}$$

horiz advection terms:

$$u\frac{\partial \xi}{\partial x} \text{ and } v\frac{\partial \xi}{\partial y} \sim U\frac{U/L}{L} = \frac{U^2}{L^2} = \boxed{10^{-10} s^{-2}}$$

vertical advection term

$$w\frac{\partial \xi}{\partial z} \sim W\frac{U/L}{H} = \frac{WU}{HL} = \frac{10^{-2}\frac{m}{s}\,10\frac{m}{s}}{10^4 m\,10^6 m} = \boxed{10^{-11} s^{-2}}$$

earth vorticity advection term

$$-v\frac{df}{dy} \sim U\beta = 10\frac{m}{s}\,10^{-11} s^{-1} m^{-1} = \boxed{10^{-10} s^{-2}}$$

Tilting Terms

$$\boxed{-\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}} \text{ and } \boxed{\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}} \sim \frac{W}{L}\frac{U}{H} = \boxed{10^{-11}\,s^{-2}}$$

baroclinic terms

$$\boxed{\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}} \text{ and } \boxed{-\frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}} \sim \frac{1}{\rho^2}\frac{\Delta\rho_{horiz}}{L}\frac{\Delta p_{horiz}}{L}$$

$$= \frac{1}{(1\,kg/m^3)^2}\,\frac{0.01\,kg/m^3}{10^6\,m}\,\frac{1000\,Pa}{10^6\,m}$$

$$= \frac{10}{10^{12}}\,\cancel{\frac{kg\,m^6}{m^3\,(kg)^2}}\,\frac{m^6}{kg^2}\,\frac{kg}{m^3}\,\frac{Pa}{m^2} \qquad Pa \rightarrow \frac{kg}{m s^2}$$

$$= \boxed{10^{-11}\,s^{-2}}$$

$$\boxed{\underset{\text{Force/area}}{1\,Pa} = \frac{N}{m^2} = \frac{kg\,m/s^2}{m^2} = \frac{kg}{m s^2}}$$

stretching term

$$(\zeta+f)\,\delta = (\zeta+f)\boxed{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}$$

Watch out!! Even though each of these terms scales as U/L, the sum should NOT be scaled by U/L because ∂u/∂x and ∂v/∂y often nearly kill each other off! (It's the 1 − 1 = ? case). How do we know? Well, for mid-latitude synoptic-scale flows the wind is pretty close to the geostrophic wind.

Since $\vec{U} \approx \vec{U}_g$, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}$

$$= \nabla_H \cdot \vec{U}_g \approx \boxed{\nabla_p \cdot \vec{U}_g = 0}$$

[see lecture 4]

So $\frac{U}{L}$ is too big a scale for $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ and 0 isn't accurate enough. Get a reasonably accurate (order of magnitude) estimate of the sum by working with the incompressibility condition:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z} \sim \frac{W}{H} = \frac{10^{-2}\,\frac{m}{s}}{10^{4}\,m} = 10^{-6}\,s^{-1}$$

[You'd get same estimate if you'd worked with anelastic mass conservation equation].

So

$$(\zeta + f)\delta = (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \sim \max\left(\underset{\uparrow \zeta}{10^{-5}\,s^{-1}};\ \underset{\uparrow f}{10^{-4}\,s^{-1}}\right) 10^{-6}\,s^{-1}$$

$$\therefore (\zeta + f)\,\delta \approx f\,\delta = \boxed{10^{-10}\,s^{-2}}$$

So, the dominant terms in the vertical vorticity equation are:

$\frac{\partial \zeta}{\partial t}$, $u\frac{\partial \zeta}{\partial x}$, $v\frac{\partial \zeta}{\partial y}$, $-(\zeta + f)\delta$ and $-\beta v$

[really just $f\delta$]

∴ approx vert vorticity eqn for synoptic-scale flows is:

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = -(\zeta + f)\,\delta - \beta v$$

$\ll$ or just neglect it

what's neglected (because it's small) in this equation?

- vert advection of vert vorticity
- tilting terms
- baroclinic terms
- friction terms (we actually didn't show they're small, but they're tiny on the synoptic scale as long as we're <u>above</u> the <u>boundary layer</u>).