Lecture 22

Approx relative vert vorticity eq^n for mid-latitude synoptic-scale flows is:

$$\boxed{\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = -(\underset{\text{or neglect it.}}{\underset{\downarrow}{\zeta}} + f)\delta - \beta v}$$

Easy to rewrite this as an eq^n for the evolution of the absolute vertical vorticity. Put $-\beta v$ term over on the left hand side and use

$\beta \equiv df/dy$:

combine

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + \left(v\frac{\partial \zeta}{\partial y} + v\frac{df}{dy}\right) = -(\zeta + f)\delta$$

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial}{\partial y}(\zeta + f) = -(\zeta + f)\delta$$

Since $f = f(y)$, can put f into the first 2 terms, get:

Since $f = f(y)$
$\frac{\partial f}{\partial x} = 0$
$\frac{\partial f}{\partial t} = 0$

$$\frac{\partial}{\partial t}(\zeta + f) + u\frac{\partial}{\partial x}(\zeta + f) + v\frac{\partial}{\partial y}(\zeta + f) = -(\zeta + f)\delta$$

In terms of absolute vert vort $\eta\ (\equiv \zeta + f)$, this becomes

$$\boxed{\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = -\eta\delta}$$

Introduce $\frac{D_h}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ (total derivative following horizontal motion of an air parcel).

∴ Approximate vert vorticity eqⁿ for mid-lat synoptic scale flows becomes:

For relative vertical vort: $\boxed{\frac{D_h \zeta}{Dt} = -\eta\delta - \beta v}$

For absolute vertical vort: $\boxed{\frac{D_h \eta}{Dt} = -\eta\delta}$

Barotropic Potential Vorticity Eqⁿ

To derive it, start with the approximate absolute vertical vort eqⁿ for synoptic-scale flows,

$$(\star) \quad \frac{D_h}{Dt}(\zeta + f) = -(\zeta + f)\,\delta$$

Consider special case where:

→ (i) Flow is incompressible, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

so $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$

→ (ii) Flow is barotropic ($\rho = \rho(p)$) ∴ no shear of geostrophic wind, i.e. no thermal wind, and

→ (iii) Flow is nearly geostrophic. So, approximate u, v and ζ by their geostrophic counterparts, u_g, v_g, and ζ_g everywhere except in the horizontal divergence! Do not approx δ by δ_g. Remember that horiz divergence of geostrophic wind is $\underline{0}$ (on an isobaric surface).

$\therefore\ \zeta \approx \zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$ and these terms are indep of height

and $\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \approx \frac{\partial}{\partial t} + \underbrace{u_g}_{\text{indep of height}}\frac{\partial}{\partial x} + \underbrace{v_g}_{\text{indep of height}}\frac{\partial}{\partial y}$

So (✱) becomes, approximately,

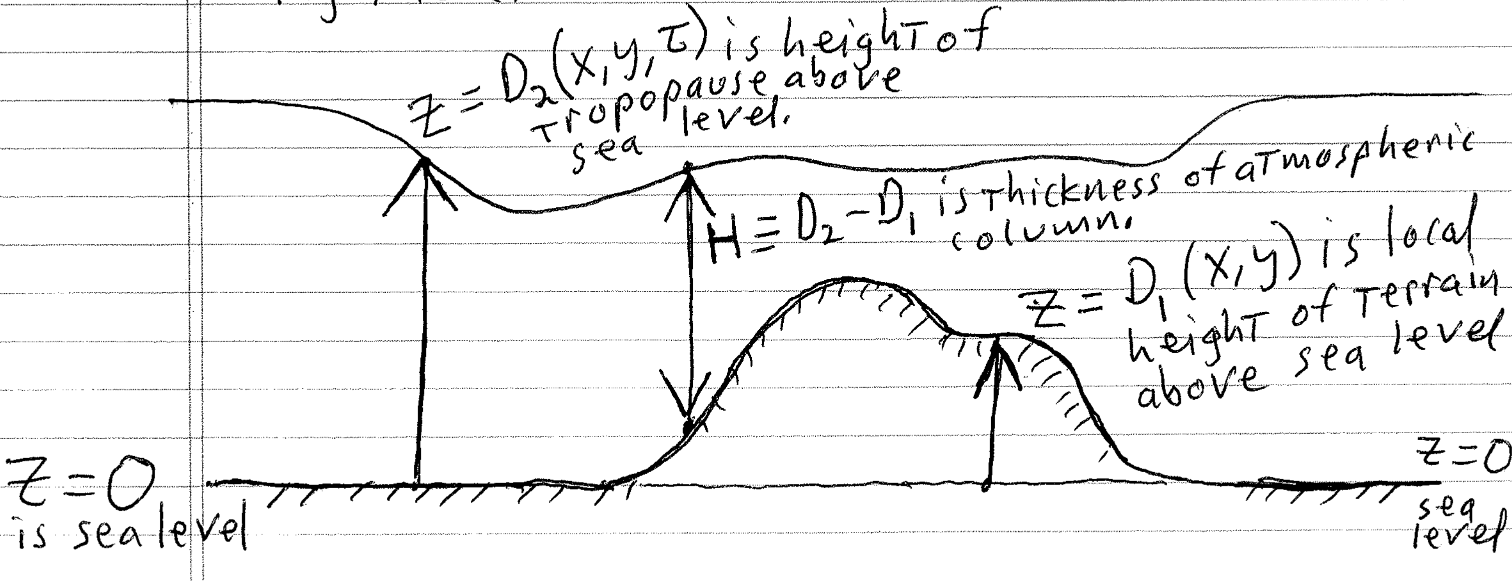
$$\frac{D_h}{Dt}(\zeta_g + f) = -(\zeta_g + f)\underbrace{\boxed{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}}_{\text{don't use geos wind in this term}} - \frac{\partial w}{\partial z}$$

where $\zeta_g + f$

and $\frac{D_h}{Dt}$ are indep of height.

→ Integrate this approx absolute vertical vort eqn from the ground (which might be mountainous!) up to the tropopause (which doesn't have to be flat!)

→ We'll omit subscript "g" but think of it as still being there.



Here's the integrated vort eqn:

$$\int_{D_1}^{D_2} \frac{D_h}{D\tau}(\zeta+f)\,dz = \int_{D_1}^{D_2} (\zeta+f)\frac{\partial w}{\partial z}\,dz$$

Pull stuff that's indep of z out of integral:

$$\frac{D_h}{D\tau}(\zeta+f)\int_{D_1}^{D_2} dz = (\zeta+f)\int_{D_1}^{D_2}\frac{\partial w}{\partial z}\,dz$$

$$(\star\star)\quad \left[\frac{D_h}{D\tau}(\zeta+f)\right](D_2-D_1) = (\zeta+f)\left[w(D_2)-w(D_1)\right]$$

$w(D_2)$ is vert velocity of parcel travelling along <u>tropopause</u>
$w(D_1)$ " " " " " " on lower surface

Since $w(z) = \frac{Dz}{D\tau}$:

$$w(D_2) = \frac{D}{D\tau}D_2(x,y,\tau) = \frac{\partial D_2}{\partial \tau} + u\frac{\partial D_2}{\partial x} + v\frac{\partial D_2}{\partial y} = \frac{D_h D_2}{D\tau}$$

$$w(D_1) = \frac{D}{D\tau}D_1(x,y) = u\frac{\partial D_1}{\partial x} + v\frac{\partial D_1}{\partial y} = \frac{D_h}{D\tau}D_1$$

So $(\star\star)$ becomes:

$$(D_2-D_1)\frac{D_h}{D\tau}(\zeta+f) = (\zeta+f)\left[\frac{D_h}{D\tau}D_2 - \frac{D_h}{D\tau}D_1\right]$$

use $H \equiv D_2 - D_1$

$$\therefore\quad H\frac{D_h}{D\tau}(\zeta+f) = (\zeta+f)\frac{D_h}{D\tau}H$$

(✡✡✡) $H \frac{D_h}{Dt}(\zeta + f) - (\zeta + f)\frac{D_h H}{Dt} = 0$

Think of it as $b\frac{Da}{Dt} - a\frac{Db}{Dt} = 0$ where $b = H$, $a = \zeta + f$

We can simplify it!

Scratch paper

$$\frac{D}{Dt}\left(\frac{a}{b}\right) = \frac{1}{b}\frac{Da}{Dt} + a\frac{D}{Dt}\frac{1}{b}$$
$$= \frac{1}{b}\frac{Da}{Dt} - \frac{a}{b^2}\frac{Db}{Dt} \quad \text{factor out } \frac{1}{b^2}$$
$$= \frac{1}{b^2}\left[b\frac{Da}{Dt} - a\frac{Db}{Dt}\right] \quad \text{mult by } b^2$$
$$\therefore \; b\frac{Da}{Dt} - a\frac{Db}{Dt} = b^2\frac{D}{Dt}\left(\frac{a}{b}\right)$$

So (✡✡✡) becomes

$$H^2\frac{D_h}{Dt}\left(\frac{\zeta + f}{H}\right) = 0 \qquad \div \text{ by } H^2$$

Get the <u>Barotropic Potential Vorticity Conservation Theorem</u> (due to Carl Gustav Rossby):

$$\boxed{\frac{D_h}{Dt}\left(\frac{\zeta + f}{H}\right) = 0} \quad \text{(for an air column)}$$

or: $$\boxed{\frac{\zeta + f}{H} = \text{const}} \quad \text{(for an air column)}$$

$\frac{\zeta + f}{H}$ is potential vorticity, the ratio:

$$\frac{\text{absolute vertical vorticity}}{\text{column thickness}}.$$

[in the above formulas, ζ is really ζ_g]

This theorem says that potential vorticity is conserved — an air column's potential vorticity does NOT change.