METR 3123, Atmospheric Dynamics II
LECTURE 23

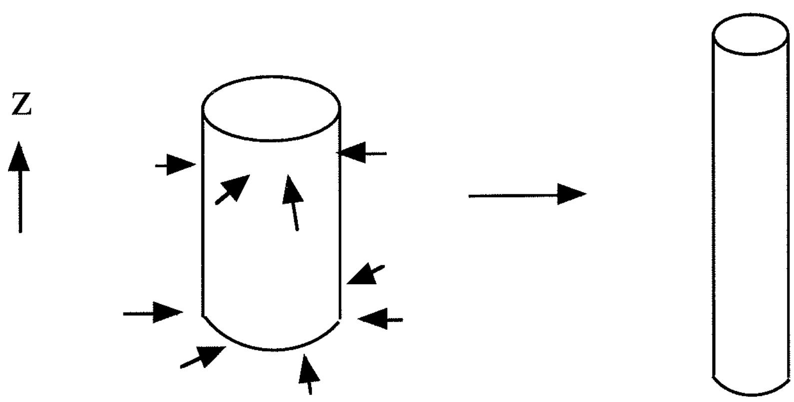
Barotropic Potential Vorticity Conservation Theorem

$$\boxed{\frac{D_h}{Dt}\left(\frac{\zeta + f}{H}\right) = 0} \quad \text{(for an air column)}$$

or: $$\boxed{\frac{\zeta + f}{H} = \text{const}} \quad \text{(for an air column)}$$

[in the above formulas, ζ is really ζ_g].

e.g. Consider an air column in the N. hemisphere with initially positive absolute vertical vorticity, $\zeta_0 + f_0 > 0$ (typical for the N. hemisphere). What happens to column's relative vert vort ζ if there's horiz convergence (vertical stretching)? Assume column latitude doesn't change (column f is const).



$\therefore$ H $\uparrow$

From the pot vort th^m, the column $\frac{\zeta + f}{H}$ is a constant, so its value at any time is equal to its initial value, $\frac{\zeta_0 + f_0}{H_0}$. So:

$\frac{\zeta + f}{H} = \frac{\zeta_0 + f_0}{H_0}$ --> $\zeta + f = \frac{H}{H_0}(\zeta_0 + f_0)$. So, since $\zeta_0 + f_0 > 0$ and $H/H_0 > 1$, the abs vert vort $\zeta + f > \zeta_0 + f_0$, that is, $\zeta + f$ increases. Since $\zeta + f \uparrow$ (with no change in f), the rel vort ζ increases. Consider two different initial states: (i) $\zeta_0 > 0$ (cylconic vort; typical of synoptic scale lows) and (ii) $\zeta_0 < 0$ (anti-cyclonic vort; typical of synoptic-scale highs):

Case (i) $\zeta_0 > 0$. Since ζ increases and ζ is initially positive, the magnitude of ζ increases. **Intensification!**

Case (ii) $\zeta_0 < 0$. Since ζ increases but ζ is initially negative, the magnitude of ζ decreases. **Weakening!**

e.g. Again consider an air column in N. hemisphere with initially positive absolute vert vort, $\zeta_0 + f_0 > 0$, but this time suppose there's horiz divergence (vertical compression). What happens to column ζ (again assuming column f doeesn't change)?

Now $H \downarrow$. Since $\zeta + f = \frac{H}{H_0}(\zeta_0 + f_0)$ and $H/H_0 < 1$ we see that $\zeta + f < \zeta_0 + f_0$, that is, $\zeta + f$ decreases. And since we're told f doesn't change, ζ decreases. Again consider the two different initial states:

Case (i) $\zeta_0 > 0$. Since ζ decreases but is initially positive, the magnitude of ζ decreases. **Weakening!**

Case (ii) $\zeta_0 < 0$. Since ζ decreases but is initially negative, the magnitude of ζ increases. **Strengthening!**

Can show stretching is more efficient at amplifying magnitude of vort than compression. So, if stretchings and compressions occur with equal frequencies and magnitudes there should be stronger cyclonic systems than anticyclonic systems -- which is generally the case (we'll examine this in a later class).

Example. At t = 0 an air column has a Coriolis parameter of $7 \times 10^{-5}\text{s}^{-1}$, a relative vorticity of 0, and it extends from a plateau 1 km above sea level (ASL) to the tropopause at 11 km ASL. Now the column moves straight eastward. After 1 day the column is over the ocean and has a relative vertical vorticity of $3.5 \times 10^{-5}\text{s}^{-1}$. What is the height of the column (tropopause height) ASL at this time?

The barotropic potential vorticity theorem says:

$$\frac{\zeta(1\text{ day}) + f(1\text{ day})}{H(1\text{ day})} = \frac{\zeta_0 + f_0}{H_0}$$

Rearrange it to get H(1 day):

$$H(1\text{ day}) = H_0 \frac{\zeta(1\text{ day}) + f(1\text{ day})}{\zeta_0 + f_0}$$

Plug in data.

$$\text{trop height (@ 1 day)} - \underbrace{\boxed{\text{base of column height (@ 1 day)}}}_{\text{0 since base of column is now at sea level!}}$$

$$= \left(11\text{ km} - 1\text{km}\right) \frac{3.5 \times 10^{-5}\text{s}^{-1} + \boxed{7 \times 10^{-5}\text{s}^{-1}}}{0 + 7 \times 10^{-5}\text{s}^{-1}}$$

(Column moved eastward, so its f didn't change)

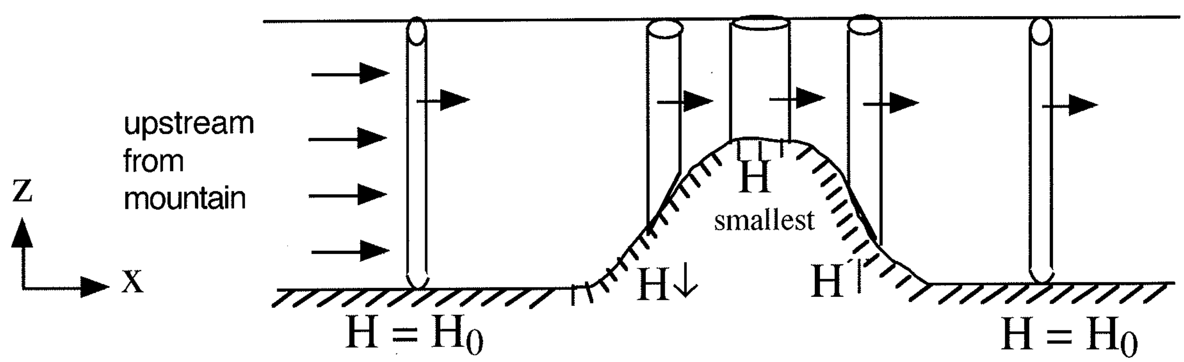
$$= (10\text{km})\left(\frac{1}{2} + 1\right) = 15\text{km}$$

$$\boxed{\text{trop height (@ 1 day)} = 15 \text{ km}}$$

Now consider a case where latitude change is important.

<u>e.g. Westerly flow over a N-S oriented mountain range.</u> Flow upstream of mountain is westerly with u = const, v = 0, w = 0.

Let's follow a column as it moves over mountain range. In xz plane the column does this:



Now deduce the column motion in xy plane.

Assume barotropic pot vort is conserved so: $\dfrac{\zeta + f}{H} = \text{const}$

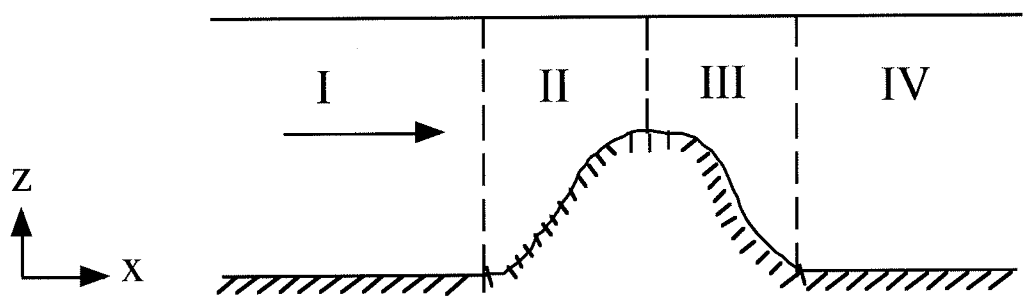
Evaluate const with initial conditions (upstream conditions). The initial rel vort ζ_0 is 0, initial H is H_0, and initial f is f_0 (corresponding to latitude ϕ_0).

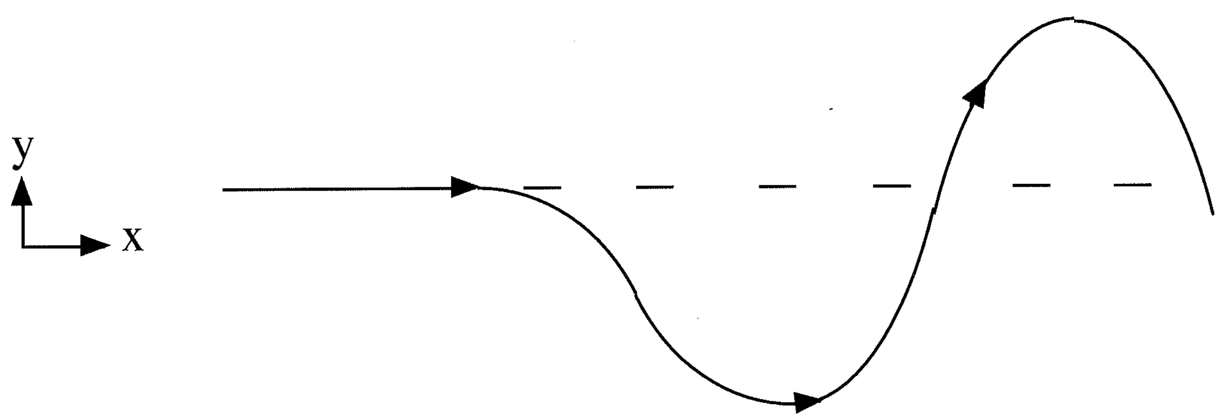
$$\text{const} = \frac{\overset{\overset{0}{\uparrow}}{\boxed{\zeta_0}} + f_0}{H_0} = \frac{f_0}{H_0}$$

$\therefore \quad \dfrac{\zeta + f}{H} = \dfrac{f_0}{H_0}$ mult by H and take f over to rhs

$\therefore \quad \zeta = f_0 \dfrac{H}{H_0} - f$ this is the column's rel vort (no matter where column is)

Determine trajectory of air column <u>assuming curvature of traj can be inferrred from sense of the vorticity</u> (not 100% legal).





I. $H = H_0, \zeta = 0, f = f_0$

II. $f = f_0$ (at first). $H \downarrow \;\therefore\; \dfrac{H}{H_0} < 1$

So $\zeta = f_0 \dfrac{H}{H_0} - f_0$, ζ becomes <u>negative</u>.

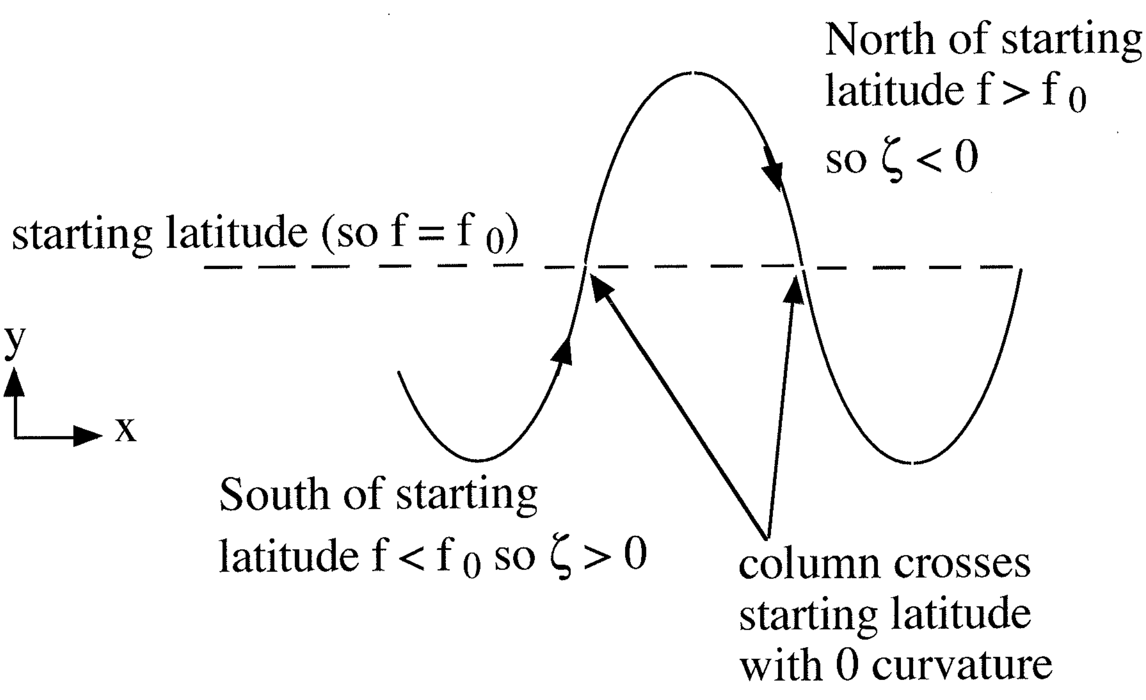
$\therefore$ column dives toward the <u>south</u> with negative curvature vort.

III. Now H $\uparrow$ while $f < f_0$ $\quad \zeta = f_0 \dfrac{H}{H_0} - f$

$\therefore$ ζ becomes positive (assume it's positive <u>curvature</u> <u>vort</u>)
Column can still move south, but with pos curvature vort.

IV. Since $H = H_0$, the formula for ζ reduces to: $\zeta = f_0 - f$. At first $f < f_0$ so $\zeta > 0$. But as column moves north, $f \uparrow$ and so $\zeta \downarrow$. This continues until column crosses it's starting latitude. At that point, $f = f_0$ and so $\zeta = 0$.

Keep tracking column. As long as terrain is flat, $\zeta = f_0 - f$, that is, $\zeta + f =$ const i.e., <u>absolute vorticity is conserved</u> (now that H const).



So, we get a wavelike trajectory downstream of mountain range.