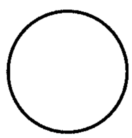
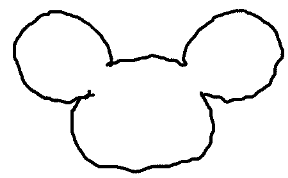
METR 3123, Atmospheric Dynamics II
LECTURE 25

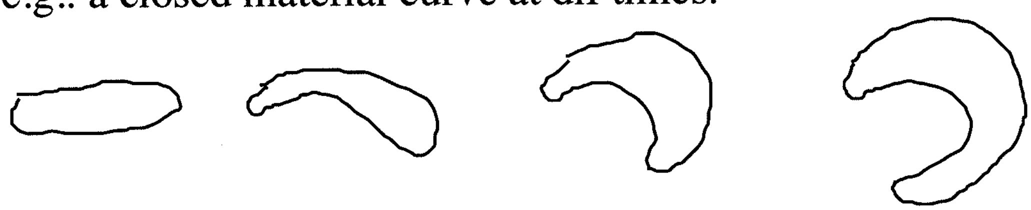
<u>Circulation Theorem</u>

Consider any arbitrary closed curve drawn in the flow:

e.g.:  or:  or: 

This curve is composed of fluid. As time goes on, this curve moves with the flow and distorts in shape, but it's always composed of same fluid elements. It's a <u>material curve</u> (curve that's always composed of same fluid elements).

e.g.: a closed material curve at dif times:



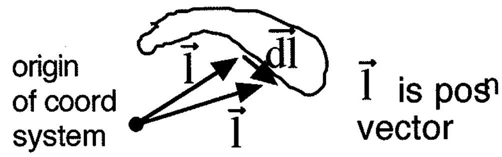
t = 0   t = 10 s   t = 20 s   t = 30 s

In general, the circulation around the closed material curve is a function of time:

$$C(t) = \oint \vec{u} \cdot \vec{dl}$$

Lets find an expression for rate of change of absolute circulation around a material curve [equivalent to rate of change of area-integrated absolute vorticity via Stokes thm]

$$C_a(t) = \oint \vec{u}_a \cdot \vec{dl}$$



$$\frac{D_a C_a}{Dt} = \frac{D_a}{Dt} \oint \vec{u}_a \cdot \vec{dl}$$

$$= \oint \boxed{\frac{D_a \vec{u}_a}{Dt}} \cdot \vec{dl} + \oint \vec{u}_a \cdot \boxed{\frac{D_a}{Dt} \vec{dl}} \rightarrow d\left(\frac{D_a \vec{l}}{Dt}\right) \rightarrow d\vec{u}_a$$

$\downarrow$

use eqn of motion (neglecting friction):

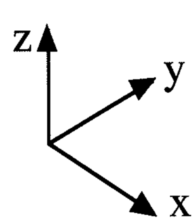
$$\frac{D_a \vec{u}_a}{Dt} = -\frac{1}{\rho} \nabla p + \vec{g}, \text{ where } \vec{g} = -g\hat{k} = -\nabla(gz) = -\nabla\Phi$$

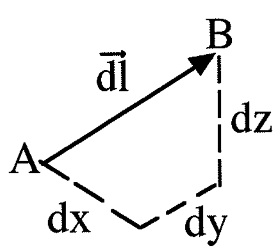
$$\therefore \quad \frac{D_a C_a}{Dt} = -\oint \frac{1}{\rho} \nabla p \cdot \vec{dl} - \oint \nabla\Phi \cdot \vec{dl} + \oint \boxed{\vec{u}_a \cdot d\vec{u}_a}$$

$\downarrow$ $\frac{DC_a}{Dt}$ since C_a is a scalar

$\downarrow$ $\frac{1}{2} d\left(\vec{u}_a \cdot \vec{u}_a\right)$

Scratch paper: How does p vary across the line element $\vec{dl}$?





$$\vec{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Use Taylor expansion for p_B about point A.

$$p_B = p_A + \left.\frac{\partial p}{\partial x}\right|_A dx + \left.\frac{\partial p}{\partial y}\right|_A dy + \left.\frac{\partial p}{\partial z}\right|_A dz$$

$$dp = p_B - p_A = \left.\frac{\partial p}{\partial x}\right|_A dx + \left.\frac{\partial p}{\partial y}\right|_A dy + \left.\frac{\partial p}{\partial z}\right|_A dz$$

$$= \nabla p \cdot \vec{dl}$$

So $\nabla p \cdot \vec{dl} = dp$

In general, change in any scalar over a directed distance $\vec{dl}$ is:

$$d(\text{scalar}) = \nabla(\text{scalar}) \cdot \vec{dl}$$

____________________________________end of scratch paper

$$\therefore \quad \frac{DC_a}{Dt} = -\oint \frac{dp}{\rho} - \boxed{\oint d\Phi} + \frac{1}{2}\boxed{\oint d\left(\vec{u}_a \cdot \vec{u}_a\right)}$$

$\downarrow$ $\Phi_{\text{endpoint}} - \Phi_{\text{startpoint}} = 0$ since end/start points are the same for a closed curve

$\downarrow$ 0 for same reason

$$\therefore \quad \boxed{\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho}}$$

Rate of change of absolute circulation is due only to baroclinic forcing.

Can show that in a barotropic fluid, $\oint \frac{dp}{\rho} = 0$. Then:

$$\boxed{\frac{DC_a}{Dt} = 0} \quad \text{Kelvin's Circulation Theorem.}$$

so C_a = const for a closed <u>material</u> curve.

Means that: "<u>Absolute circulation is conserved following fluid motion in a frictionless barotropic fluid</u>"

Now lets relate absolute circulation C_a to relative circ C:

$$C_a = \oint \vec{u}_a \cdot \vec{dl} = \oint \left(\vec{u} + \vec{\Omega} \times \boxed{\vec{r}}\right) \cdot \vec{dl}$$

↓ rel velocity ↓ pos^n vector (same as $\vec{l}$)

$$= \oint \vec{u} \cdot \vec{dl} + \oint \left(\vec{\Omega} \times \vec{r}\right) \cdot \vec{dl}$$

<u>scratch paper</u>

Lets rewrite $\oint \left(\vec{\Omega} \times \vec{r}\right) \cdot \vec{dl}$ using Stokes thm.

Stokes thm: $\oint \vec{V} \cdot \vec{dl} = \iint \hat{n} \cdot \left(\nabla \times \vec{V}\right) dA$ for <u>any</u> vector $\vec{V}$.

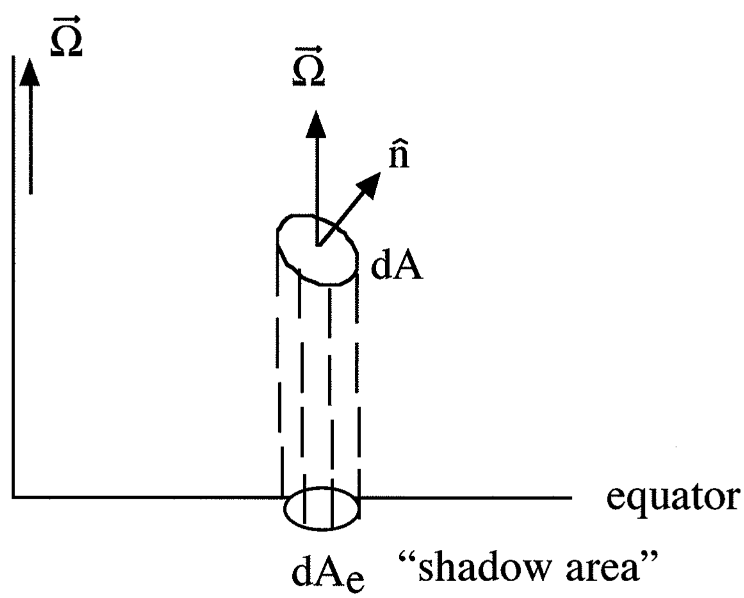
Let $\vec{V} \equiv \vec{\Omega} \times \vec{r}$. Then: $\oint \left(\vec{\Omega} \times \vec{r}\right) \cdot \vec{dl} = \iint \hat{n} \cdot \left[\nabla \times \left(\vec{\Omega} \times \vec{r}\right)\right] dA$

$$\therefore\ C_a = C + \iint \hat{n} \cdot \boxed{\left[\nabla \times \left(\vec{\Omega} \times \vec{r}\right)\right]} dA$$

$\downarrow$ rel circ $\qquad$ $\downarrow$ $= 2\vec{\Omega}$ (done in prob set)

$$\therefore\quad C_a = C + \iint 2\vec{\Omega} \cdot \hat{n}\, dA$$

$\hat{n}\, dA$ is tiny directed area element. Let dA_e be the <u>projection</u> of $\hat{n}\, dA$ onto the equatorial plane (plane $\perp$ to $\vec{\Omega}$):



$$dA_e \equiv \frac{\vec{\Omega}}{\Omega} \cdot \hat{n}\, dA \quad [\frac{\vec{\Omega}}{\Omega} \text{ and } \hat{n} \text{ are unit vectors}]$$

mult by 2Ω

$$2\Omega\, dA_e = 2\vec{\Omega} \cdot \hat{n}\, dA$$

$$\therefore \quad C_a = C + \iint 2\Omega dA_e$$ Evaluate integral, get:

$$\boxed{C_a = C + 2\Omega A_e}$$

absolute circ = relative circ + earth vort x total projected area

So Kelvin's circulation theorem says that for closed material curve moving in a frictionless, barotropic flow,

(absolute version): $\frac{DC_a}{Dt} = 0$ or $C_a = \text{const}$

(relative version): $\frac{DC}{Dt} + 2\Omega\frac{DA_e}{Dt} = 0$ or $C + 2\Omega A_e = \text{const}$

where A_e is the projection of the area enclosed by the material curve on the equatorial plane.