METR 3123, Atmospheric Dynamics II
LECTURE 26

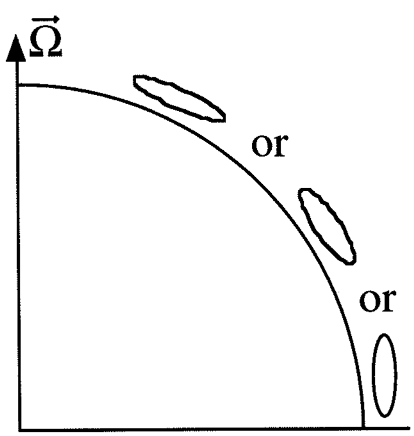
Circulation Theorem (continued)

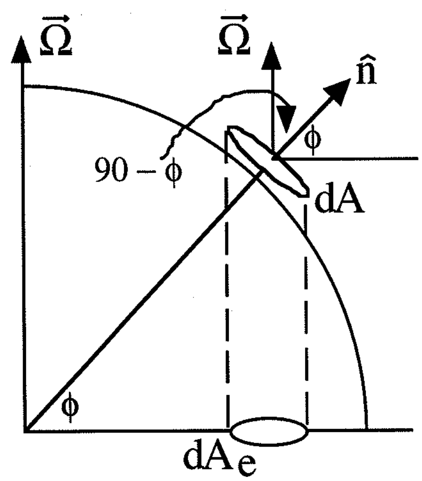
Kelvin's circulation theorem says that for closed material curve moving in a frictionless, barotropic flow,

(absolute version): $\frac{DC_a}{Dt} = 0 \quad \text{or} \quad C_a = \text{const}$

(relative version): $\frac{DC}{Dt} + 2\Omega\frac{DA_e}{Dt} = 0 \quad \text{or} \quad C + 2\Omega A_e = \text{const}$

Consider special case where material curve and enclosed area is horizontal (tangent to earth's sfc), so $\hat{n} = \hat{k}$).





$$dA_e = dA\, \hat{n} \cdot \frac{\vec{\Omega}}{\Omega} = dA \left|\hat{n}\right| \left|\frac{\vec{\Omega}}{\Omega}\right| \cos(90 - \phi) = dA \sin\phi$$

$$\underset{1}{\downarrow} \quad \underset{1}{\downarrow} \quad \underset{\cos 90 \cos\phi + \sin 90 \sin\phi}{\downarrow}$$

So total projected area is:

$$A_e = \iint dA_e = \iint \sin\phi \, dA \qquad \text{insert } \frac{A}{A} \text{ on rhs}$$

$$= A\left(\frac{1}{A}\iint \sin\phi \, dA\right) \qquad \text{stuff in ( ) is an ave}$$

$$\therefore \quad A_e = A\langle\sin\phi\rangle \quad \text{(for a material curve that stays horizontal)}$$

↓
ave of $\sin\phi$

So for this special material curve, the circulation thm becomes:

$$C(t) + 2\Omega A(t)\langle\sin\phi(t)\rangle = \text{const}$$

Use initial conditions to evaluate the const.

$$C(t_0) + 2\Omega A(t_0)\langle\sin\phi(t_0)\rangle = \text{const}$$

$$\therefore \quad C + 2\Omega A\langle\sin\phi\rangle = C(t_0) + 2\Omega A(t_0)\langle\sin\phi(t_0)\rangle$$

e.g., if curve moves <u>north</u> keeping its area A constant then $\phi\uparrow$ $\therefore$ $\sin\phi\uparrow$ $\therefore$ $C\downarrow$

e.g., if area enclosed by curve increases while mean latitude is constant, then $A\uparrow$ $\left(\text{and } A_e\uparrow\right)$ $\therefore$ $C\downarrow$

e.g. Consider a horiz circular region of air of radius 100 km centered at equator. The air is initially at rest (w.r.t. earth). Suppose the air is forced to move north, preserving its area.

What is the circulation around the circumference when the circular air mass reaches north pole? (neglect baroclinic effects).

Use Kelvin's th^m: $C(t) + 2\Omega A_e(t) = \text{const}$

Or, since area is tangent to earth's sfc:

$$C(t) + 2\Omega A(t)\sin\phi(t) = \text{const}$$

Use initial conditions to evaluate the const. Air is initially resting so $C(t_0) = 0$. Initially at equator so $\langle \sin\phi(t_0) \rangle = 0$

$$\therefore \quad 0 + 2\Omega A(t_0) \times 0 = \text{const} \qquad \therefore \text{const} = 0$$

$$\therefore \quad C(t) + 2\Omega A(t)\sin\phi(t) = 0$$

$$\therefore \quad C(t) = -2\Omega A(t)\sin\phi(t)$$

So, air mass reaches north pole with circulation:

$$C = -2\Omega\pi r^2 \sin 90 = -2\left(\frac{2\pi}{24 \times 3600\text{s}}\right)\pi\,(100{,}000\text{m})^2$$

$$= -4.63 \times 10^6\,\text{m}^2/\text{s} \quad \text{[anticyclonic relative circ]}$$

What is the average velocity comp tangent to the circle, $\bar{v}_\theta$?

$$\bar{v}_\theta = \frac{1}{2\pi r}\oint \vec{u}\cdot d\vec{l} = \frac{C}{2\pi r}$$

$$= \frac{-4.63 \times 10^6\,\text{m}^2/\text{s}}{2\pi \times 100{,}000\text{m}} = -7.4\ \text{m/s} \quad \text{[anticyclonic flow]}$$

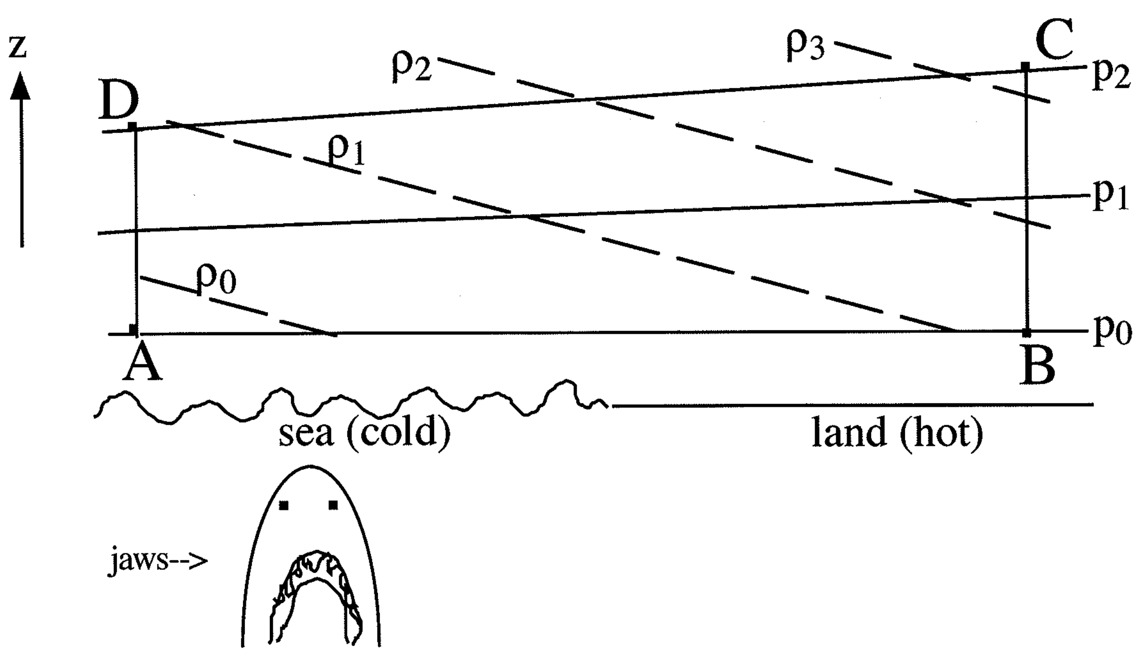
If baroclinic effects are important, keep baroclinic term in Circulation Th^{m}. Get:

Circulation Theorem for abs circ: $$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho}$$

Circulation Theorem for rel circ: $$\boxed{\frac{DC}{Dt} + 2\Omega \frac{DA_e}{Dt} = -\oint \frac{dp}{\rho}}$$

Bjerknes Circulation Theorem

e.g. sea-breeze during morning/early afternoon in the summer. [Occurs near oceans or big lakes. Also "inland sea breeze" near rain-cooled land areas. Related circulations induced near ploughed/harvested fields (winter wheat effect). Katabatic flows along cooled slopes/glaciers.]



During morning/day, land heats up. Gets hotter than sea.

Consider closed contour composed of isobaric sfc p_0 (from A to B), vertical side from B to C, isobaric sfc p_2 (from C to D) and the vertical side from D to A. This will be a convenient arrangement because $dp = 0$ all along an isobaric surface (so the integrals along those surfaces drop out)!