LECTURE 29

The Planetary Boundary Layer (continued)

Parameterization of the PBL in NWP/climate models

Numerical weather prediction (NWP) and climate models solve the equations of motion, thermal energy, and equations for moisture species on a computer. The equations are mostly prognostic (predictive; contain a local derivative $\partial/\partial t$). The solution consists of integrating the governing equations with respect to time to get the dependent variables (u, v, w, T, etc) on fixed grid points or grid cells (or for a set of Fourier modes if a "spectral" code is used).

Because of the relatively large grid spacing of most NWP/climate models most eddies in the PBL are too small to be resolved. NWP/climate models incorporate the effects of these eddies into their solution procedure by calling computer subroutines where these effects are parameterized. The parameterizations are developed theoretically from the following steps (isolated from main NWP/climate model code):

(1) Governing equations considered are not the same as the equations in the original NWP/climate model. They're usually expressed in a **form appropriate for mesoscale modelling**, and usually the **Boussinesq approximation is made**.

(2) The governing equations are "averaged". The technique is called **Reynolds averaging**, and there are several variants.

(3) The Reynolds-averaged equations contain unknown terms arising from unresolved (small-scale) motions including turbulent motions. Parameterization consists of relating these unknown terms to averaged variables. The theories for how to do it all have side effects. Parameterization is a necessary but dirty business.

In the next few classes we'll look in detail at (1) and (2), and give an example of (3).

Large Eddy Simulation (LES) of the PBL

By using the tools of Large Eddy Simulation (LES), you **can** numerically model many of the eddies in the PBL explicitly using high spatial resolution (say ~10 m grid spacing). But you are generally limited by computer resources to horizontal domains of a few 10s of km (instead of 1000s of km as in larger scale NWP).

The governing equations in LES are usually expressed in a **form appropriate for mesoscale modelling**, and usually the **Boussinesq approximation is made**.

The governing equations in LES are averaged (with an "LES filter") but the nature of the averaging is different from Reynolds-averaging. In LES much of the energy in the turbulent motion (eddies) is resolved, so you don't want to average it out. The LES filter is designed to remove scales associated with high-frequency (small scale features), and leave behind features that can be resolved.

Direct Numerical Simulation (DNS) of the PBL

In Direct Numerical Simulation (DNS), the governing equations are solved with a resolution high enough to resolve the smallest eddies (a few mm), but on very small domains (a meter or so). Obviously you can't model a realistic atmospheric PBL this way, but you can model a "virtual PBL", one associated with weaker forcings.

The governing equations in DNS are usually expressed in a **form appropriate for mesoscale modelling**, and usually the **Boussinesq approximation is made**.

The governing equations are solved without doing any averaging/filtering.

Equations of motion for mesoscale dynamics

The equations of motion appropriate for many mesoscale applications are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f\,v + \nu\nabla^2 u\,, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - f\,u + \nu\nabla^2 v\,, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\nabla^2 w\,. \tag{3}$$

Usually, however, (1)-(3) are rewritten in a form where the pressure p and density ρ fields have been split into **reference** (also known as base-state) and **perturbation** (also known as deviation) values.

A **reference atmosphere** is a hypothetical **motionless** atmosphere in which the **density** $\rho_0(z)$ **is horizontally homogeneous** (indep of x and y) and **indep of time**, and the **pressure** $p_0(z)$ **satisfies the hydrostatic equation** with density ρ_0: $dp_0/dz = -\rho_0\,g$. One can get $\rho_0(z)$ from a sounding but that's not the only option.

The actual density ρ can be decomposed into two parts: the reference density ρ_0 and the part of the density that's not the reference density, the perturbation density ρ':

$$\underbrace{\rho(x,y,z,t)}_{\text{density}} \quad = \quad \underbrace{\rho_0(z)}_{\text{ref density}} \quad + \quad \underbrace{\rho'(x,y,z,t)}_{\text{perturbation density}}$$

Rearrange it to get an expression (really the definition) for the perturbation density:

$$\boxed{\rho'(x,y,z,t) \quad \equiv \quad \rho(x,y,z,t) \quad - \quad \rho_0(z)}\,. \tag{4}$$

Similary, we define the perturbation pressure p' as:

$$\boxed{p'(x,y,z,t) \equiv p(x,y,z,t) - p_0(z)}. \tag{5}$$

In almost all meteorological applications, the magnitude of ρ' is very small, about 0.01-0.03 times ρ (a few percent of ρ). So (4) indicates that ρ is nearly the same as ρ_0. So $\rho' << \rho$ and $\rho' << \rho_0$, and therefore:

$$\frac{\rho'}{\rho} << 1, \quad \text{and} \quad \frac{\rho'}{\rho_0} << 1 \quad \text{(in terms of magnitudes)}$$

Similarly, the magnitude of p' is very small, often 0.01 times p or even less. So (5) indicates that p is nearly the same as p_0. So $p' << p$ and $p' << p_0$, and therefore:

$$\frac{p'}{p} << 1, \quad \text{and} \quad \frac{p'}{p_0} << 1 \quad \text{(in terms of magnitudes)}$$

Despite their smallness, ρ' and p' can be very important dynamically. So don't throw them completely out! But you can take advantage of their smallness to rewrite (1)-(3). For the pressure gradient force (pgf) term in (1) consider the following:

$$\frac{\partial p}{\partial x} = \boxed{\frac{\partial p_0}{\partial x}}^{\to 0} + \frac{\partial p'}{\partial x} = \frac{\partial p'}{\partial x}. \qquad [\frac{\partial p_0}{\partial x} = 0 \text{ since } p_0 \text{ is a function of } z \text{ (only)}]$$

$$\frac{1}{\rho} = \frac{1}{\rho_0 + \rho'} = \frac{1}{\rho_0\left(1 + \frac{\rho'}{\rho_0}\right)} = \frac{1}{\rho_0}\left(1 + \frac{\rho'}{\rho_0}\right)^{-1}$$

$$\approx \frac{1}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right) \quad \text{from the binomial approximation}$$

$$\approx \frac{1}{\rho_0}. \qquad \text{since } \frac{\rho'}{\rho_0} << 1$$

So, the x-component pgf can be approximated as:

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x} \approx -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}}. \quad (6)$$

Similarly, the y-component pgf can be approximated as:

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial y} \approx -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}}. \quad (7)$$

The right hand sides of (6) and (7) are sometimes called the x and y components of the **perturbation pgf**.

For the vertical equation of motion (3) we combine the vertical pgf and gravity:

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = -\frac{1}{\rho}\left(\frac{\partial p}{\partial z} + \rho g\right) = -\frac{1}{\rho}\left(\frac{\partial p_0}{\partial z} + \frac{\partial p'}{\partial z} + \rho_0 g + \rho' g\right) \qquad \text{use } \frac{dp_0}{dz} = -\rho_0 g$$

$$= -\frac{1}{\rho}\left(-\rho_0 g + \frac{\partial p'}{\partial z} + \rho_0 g + \rho' g\right) = -\frac{1}{\rho}\left(\frac{\partial p'}{\partial z} + \rho' g\right) \qquad \text{use} \frac{1}{\rho} \approx \frac{1}{\rho_0}$$

$$\approx -\frac{1}{\rho_0}\left(\frac{\partial p'}{\partial z} + \rho' g\right) = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0} g .$$

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial z} \quad - g \quad \approx \quad -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} \quad -\frac{\rho'}{\rho_0} g} \quad (8)$$

vertical pgf	gravity	vertical perturbation pgf	buoyancy

Reality checks on the minus sign in buoyancy:

Where air is **colder** than in the reference atmosphere, the density is larger than the reference density ($\rho > \rho_0$), and (4) shows that ρ' is **positive**. In this case the buoyancy $-\dfrac{\rho'}{\rho_0}g$ is **negative** and represents a downward force. Checks out!

Where air is **warmer** than in the reference atmosphere, the density is smaller than the reference density ($\rho < \rho_0$), and (4) shows that ρ' is **negative** (there's no such thing as a negative density but ρ' isn't a density, it's a difference between densities). So $-\dfrac{\rho'}{\rho_0}g$ is **positive** and represents an upward force. Checks out!

So, using (6)-(8) we can rewrite (1)-(3) as:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}+f\,v+\nu\nabla^2 u, \tag{9}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p'}{\partial y}-f\,u+\nu\nabla^2 v, \tag{10}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p'}{\partial z}-\frac{\rho'}{\rho_0}g+\nu\nabla^2 w. \tag{11}$$