Lecture 3 (1)

Review of geostrophic wind

Consider the horizontal equations of motion:

x-comp eq^n of motion : $\boxed{\frac{Du}{Dt}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x}} \boxed{+ fv}$

y-comp eq^n of motion : $\boxed{\frac{Dv}{Dt}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial y}} \boxed{- fu}$ → Coriolis force terms

acceleration terms; horiz pressure gradient force (p.g.f.) term

where u is the x-comp of the velocity
v " " y-comp " " "

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$f \equiv 2\Omega \sin\phi$, where ϕ is latitude

↑ Coriolis parameter

(so $f>0$ in N. hemisphere, and $f<0$ in S. hemisphere).

For the hypothetical case of a flow that's NOT accelerating, $\frac{Du}{Dt} = 0$ and $\frac{Dv}{Dt} = 0$.

Then the equations of motion reduce to:

(1) $0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv_g$

(2) $0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu_g$

introduce subscript "g" for geostrophic

This is a two-way balance btw horiz pressure gradient force and Coriolis force. It's a Geostrophic Balance!

Let's express this balance as a single vector eq^n.

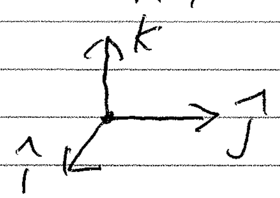
Take $\hat{i}$ times (1) + $\hat{j}$ times (2):

$$(3) \quad 0 = -\frac{1}{\rho}\left(\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y}\right) + f\left(v_g\hat{i} - u_g\hat{j}\right)$$

$\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y}$ is ∇p (well, horizontal gradient of p).

What about $v_g\hat{i} - u_g\hat{j}$? Can relate it to the horizontal velocity vector $\vec{U}_g = u_g\hat{i} + v_g\hat{j}$.

How so?



From right hand rule:

$\hat{i} = \hat{j}\times\hat{k}$ and $\hat{j} = \hat{k}\times\hat{i}$

So
$$\begin{aligned}
v_g\hat{i} - u_g\hat{j} &= v_g\hat{j}\times\hat{k} - u_g(\hat{k}\times\hat{i}) \\
&= -v_g\hat{k}\times\hat{j} - u_g\hat{k}\times\hat{i} \\
&= -\hat{k}\times\left(v_g\hat{j} + u_g\hat{i}\right) \\
&= -\hat{k}\times\left(u_g\hat{i} + v_g\hat{j}\right) \\
&= -\hat{k}\times\vec{U}_g
\end{aligned}$$

So (3) becomes:

$$(4) \quad \boxed{0 = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{horiz pgf}} \underbrace{- f\hat{k}\times\vec{U}_g}_{\text{Coriolis force}}}$$

Vector form of geostrophic balance eq^n

Can get components of geostrophic wind from (1) and (2) as:

$$(5) \quad U_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y}$$

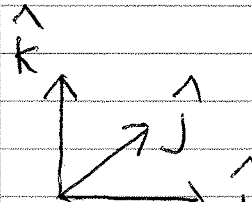
Careful with the signs!

$$(6) \quad v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x}$$

<u>Can get the vector form of the geostrophic wind $\vec{U}_g$ from (5) and (6)</u> <u>or</u> <u>just from (4).</u>
<u>Let's do it both ways:</u>

Take $\hat{i}$ times (5) + $\hat{j}$ times (6):

$$(7) \quad \underbrace{\hat{i}\, U_g + \hat{j}\, v_g}_{\text{it's } \vec{U}_g!} = \frac{1}{\rho f}\underbrace{\left(-\hat{i}\frac{\partial p}{\partial y} + \hat{j}\frac{\partial p}{\partial x}\right)}_{\text{This is... uh... well, we can relate it to } \nabla p.}$$



$$\hat{i} = \hat{j} \times \hat{k} \quad \text{and} \quad \hat{j} = \hat{k} \times \hat{i}$$

$$\therefore \; -\hat{i}\frac{\partial p}{\partial y} + \hat{j}\frac{\partial p}{\partial x} = -\hat{j}\times\hat{k}\frac{\partial p}{\partial y} + \hat{k}\times\hat{i}\frac{\partial p}{\partial x}$$

$\hat{j}\times\hat{k} = -\hat{k}\times\hat{j}$

$$= \hat{k}\times\hat{j}\frac{\partial p}{\partial y} + \hat{k}\times\hat{i}\frac{\partial p}{\partial x}$$

$$= \hat{k}\times\left(\hat{j}\frac{\partial p}{\partial y} + \hat{i}\frac{\partial p}{\partial x}\right)$$

$$= \hat{k}\times\left(\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y}\right)$$

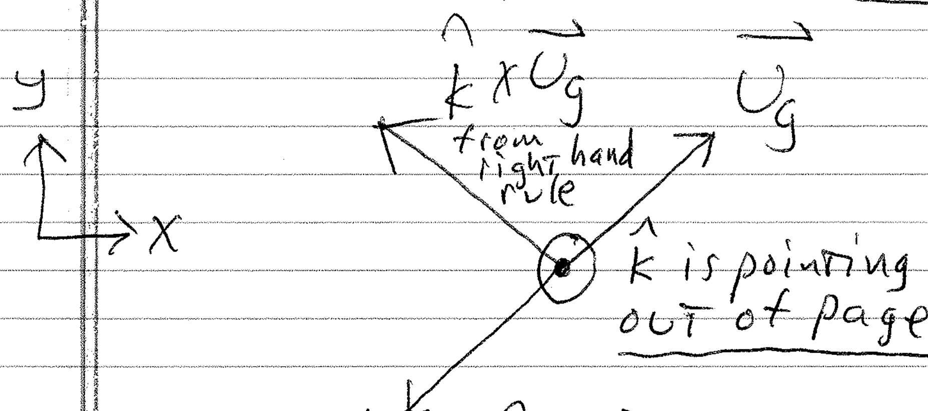
$$= \hat{k}\times\nabla p$$

So (7) becomes:

$$(8) \quad \vec{U}_g = \frac{1}{\rho f} \hat{k} \times \nabla p$$

Can get same result by taking $\hat{k} \times (4)$:

$$(9) \quad 0 = \hat{k} \times \left(-\frac{1}{\rho} \nabla p\right) - f \underbrace{\boxed{\hat{k} \times (\hat{k} \times \vec{U}_g)}}_{\text{investigate}}$$



$\hat{k} \times (\hat{k} \times \vec{U}_g)$ points <u>opposite</u> $\vec{U}_g$

from an second pass of right hand rule.

what is magnitude of $\hat{k} \times (\hat{k} \times U_g)$?

$$|\hat{k} \times (\hat{k} \times \vec{U}_g| = \underbrace{|\hat{k}|}_{1} \, |\hat{k} \times \vec{U}_g| \underbrace{\sin 90^\circ}_{1}$$

$$= |\hat{k} \times \vec{U}_g|$$

$$= |\hat{k}| |\vec{U}_g| \sin 90^\circ$$

$$= |\vec{U}_g|$$

So $\hat{k} \times (\hat{k} \times \vec{U_g})$ has magnitude $|\vec{U_g}|$

and it has direction opposite of $\vec{U_g}$, so direction of $-\vec{U_g}$. Unit vector in that direction is

$$\frac{-\vec{U_g}}{|-\vec{U_g}|} = \frac{-\vec{U_g}}{|\vec{U_g}|}$$

→ Recall that any vector is equal to it's magnitude times unit vector in its own direction. So:

$$\hat{k} \times (\hat{k} \times \vec{U_g}) = \underbrace{\frac{-\vec{U_g}}{|\vec{U_g}|}}_{\text{unit vector in its own direction}} \underbrace{|\vec{U_g}|}_{\text{magnitude}} = -\vec{U_g}$$

So (9) becomes

$$0 = \hat{k} \times \left(-\frac{1}{\rho}\nabla p\right) - f\left(-\vec{U_g}\right)$$

÷ by f then rearrange it, get:

$$\therefore \boxed{\vec{U_g} = \frac{1}{\rho f}\, \hat{k} \times \nabla p}$$

which is what we had earlier in (8).

Lets examine graphically what the geostrophic balance eq^n (4) and equation (8) for the geostrophic wind are trying to tell us.

Consider a geostrophically-balanced flow in which the p field on a constant height surface looks like:

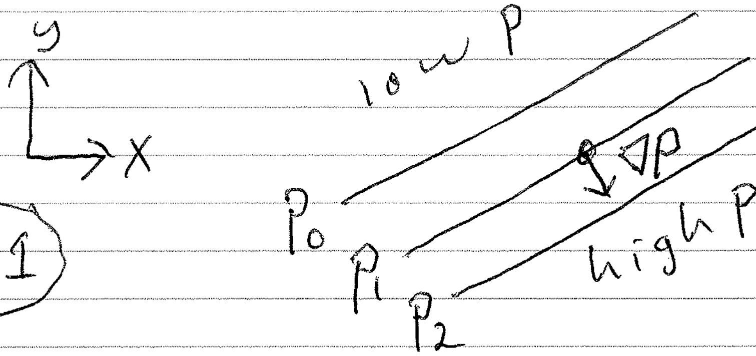


Fig. 1

pressure gradient points towards higher pressure

So the horizontal p.g.f. looks like:

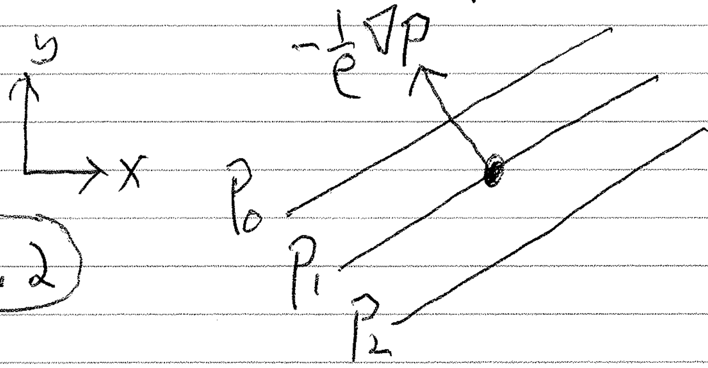
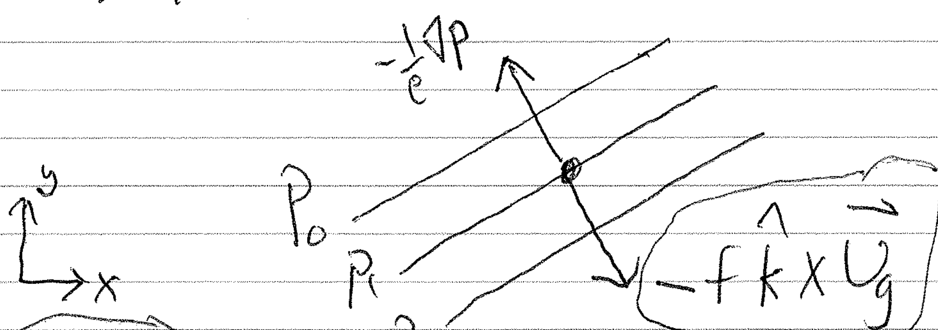


Fig. 2

The p.g.f. points opposite the pressure gradient!

So, from (4) we can infer that the Coriolis force must look like:



(Coriolis force vector)

Fig. 3

it is equal and opposite to the p.g.f.

which way is the geostrophic wind pointing? Let's consider case where we're in the Northern hemisphere (so $f > 0$).

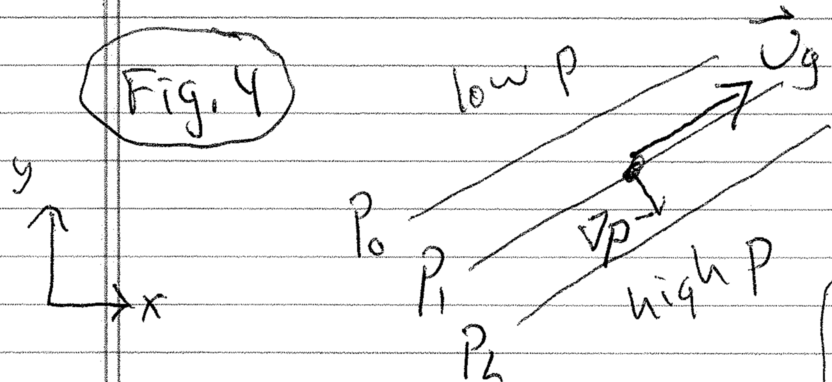
Eq^{n} (8) says $\vec{U_g} = \frac{1}{\rho f} \hat{k} \times \nabla p$

ρ: always pos

f: pos in N. hemisphere (as in our example)

So $\vec{U_g}$ has the <u>direction</u> of $\hat{k} \times \nabla p$:

Fig. 4



Note that $\vec{U_g}$ blows parallel to isobars.

With geos wind at your back, low p is toward your left (in N. hemisphere.)

As a reality check, does the direction of $\vec{U_g}$ above give us the correct Coriolis force direction? (as indicated in Fig. 3)

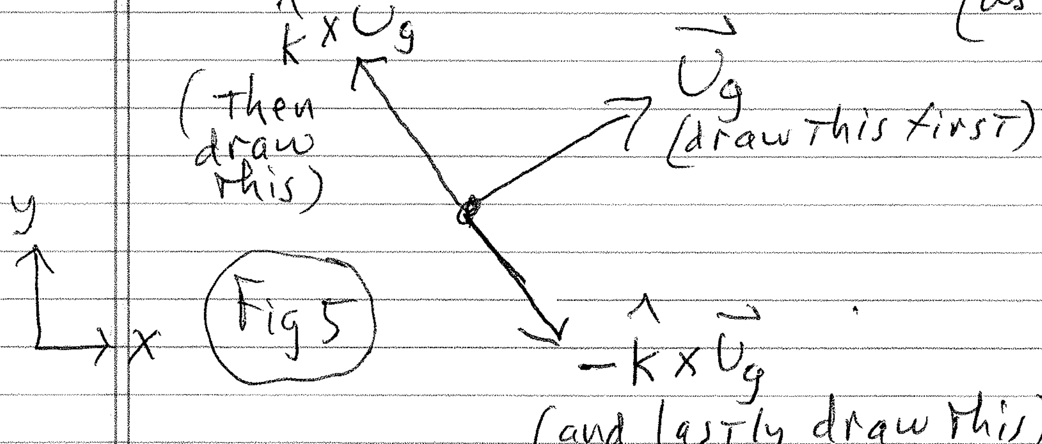


Fig 5

and since $f>0$, the Coriolis force $-f\hat{k} \times \vec{U_g}$ also appears in the direction of $-\hat{k} \times \vec{U_g}$

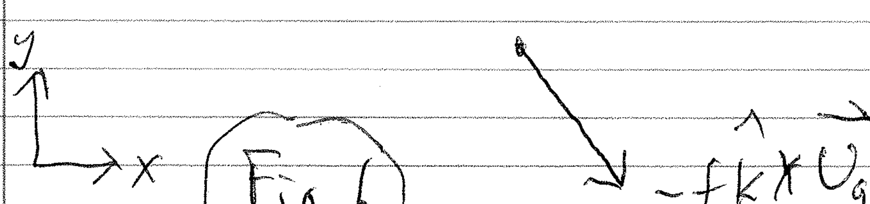


Fig. 6

yes, it points same way as in Fig 3.