LECTURE 30
The Planetary Boundary Layer (continued)

Boussinesq approximation

The Boussinesq approximation is appropriate for flows with density variations that aren't too great. Such flows have small vertical length scales (~1 km). The Boussinesq approximation is really a set of approximations:

(i) ν and κ are held constant.

(ii) exact mass conservation equation $\frac{D\rho}{Dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$ is approximated by the <u>incompressibility condition</u>, $\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0}$.

(iii) The reference density $\rho_0(z)$ in the **denominators of the pgf and buoyancy terms** in the equations of motion (9)-(11) is replaced by a constant (for now, call it ρ_c). Overall, this means that density is treated as a constant everywhere **except where it's coupled with gravity** (i.e., in the buoyancy term).

So, under the Boussinesq approximation, the equations of motion (9)-(11) become:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_c}\frac{\partial p'}{\partial x}+fv+\nu\nabla^2 u, \tag{12}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho_c}\frac{\partial p'}{\partial y}-fu+\nu\nabla^2 v, \tag{13}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_c}\frac{\partial p'}{\partial z}-\frac{\rho'}{\rho_c}g+\nu\nabla^2 w, \tag{14}$$

where we still treat the perturbation quantities as $\rho' \equiv \rho - \rho_0(z),\ p' \equiv p - p_0(z)$.

Warnings!!!!! Alarm bells!!! Ding!!! Ding!!! Ding!!!

If you get confused reading about the Boussinesq approximation in text books or journal articles, it may be because:

1. Some authors get the Boussinesq approximation wrong, and have a height-varying density in the denominators of the perturbation pgf and buoyancy.

2. Some authors have fallen into a bad notational habit that keeps propagating over the years. They work with the perturbation pgf and buoyancy in the form $-\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0} g$ and state that ρ_0 is constant, but continue to treat ρ_0 as a function of z in $\rho' \equiv \rho - \rho_0(z)$ and in the calculation of $p_0(z)$, which appears in $p' \equiv p - p_0(z)$. So the same symbol ρ_0 represents two different things in the same term -- a constant when it's in the denominator, and a function of *z* in the numerator. As long as you use "ρ_0" consistently in this inconsistent way, it's correct. But it is bizarre and confusing.

3. Some authors treat the density in the reference state as constant, $\rho_0 = \text{const}$. So the perturbation pgf and buoyancy appear as $-\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0} g$, with the denominators being constant (good!). However, ρ' is then defined as a deviation from a constant, $\rho' \equiv \rho - \rho_0$, and p' is obtained as $p' \equiv p - p_0(z)$ where $p_0(z)$ is based on constant ρ_0. So these perturbations are different from the perturbations we work with. Remarkably, however, the system of equations with the perturbations defined either way yields the same solutions for *u*, *v*, *w*, ρ, and *p* (the equivalence isn't obvious).

A further approximation (beyond Boussinesq) is often made to the vertical equation of motion

Often it's convenient to work with buoyancy expressed in terms of perturbation potential temperature $\theta' \equiv \theta - \theta_0(z)$ [deviation of potential temperature θ from the potential temperature in the reference state, $\theta_0(z)$],

$$-\frac{\rho'}{\rho_c}g = -\frac{\rho - \rho_0(z)}{\rho_c}g \approx \frac{\theta - \theta_0(z)}{\theta_c}g = \boxed{\frac{\theta'}{\theta_c}g}, \tag{15}$$

where θ_c is a constant value of potential temperature. Note that while there's a minus sign in $-\frac{\rho'}{\rho_c}g$, there is no minus sign in $\frac{\theta'}{\theta_c}g$. [Where air is warmer than in reference atmosphere, θ' is positive, however, as discussed in lecture 29, ρ' is negative). One of the upcoming homework problems will show where (15) comes from.

Ensemble averaging

Consider a turbulent flow being modelled in a wind tunnel. Let $\varphi(\vec{r},t)$ represent any variable (*u*, *v*, *w*, *T*, etc) at location $\vec{r}$ at time *t*. If we run one experiment, and examine $\varphi(\vec{r},t)$ at a specific location and time, we observe the value φ_1 (1 means experiment 1). If we run a second experiment under the "same" conditions, and examine $\varphi(\vec{r},t)$ at the same location and time as before, we observe the value φ_2. Because the flow is turbulent, φ_2 will differ from φ_1 (there's such extreme sensitivity to the slightest differences in initial conditions and other aspects of the way the experiment is run in the real world, that the values can't possibly be the same). We run more experiments and observe values φ_3, φ_4, φ_5, and so on. If we want a single estimate for φ that best characterizes the average of φ at that location and time, we're led to the idea of an **ensemble average**:

$$\overline{\varphi}(\vec{r},t) = \lim_{N\to\infty} \frac{\varphi_1(\vec{r},t) \ + \ \dots \ + \ \varphi_N(\vec{r},t)}{N}. \tag{16}$$

We can then decompose the actual $\varphi(\vec{r},t)$ into an ensemble mean $\overline{\varphi}$ and fluctuating (turbulent) part φ',

$$\varphi(\vec{r},t) = \overline{\varphi}(\vec{r},t) + \varphi'(\vec{r},t). \tag{17}$$

In other words, we define φ' by:

$$\varphi'(\vec{r},t) = \varphi(\vec{r},t) - \overline{\varphi}(\vec{r},t). \tag{18}$$

Note that this prime ($'$) is completely different from the prime ($'$) used to discuss perturbation density/pressure.