LECTURE 31

The Planetary Boundary Layer (continued)

Ensemble averaging (continued)

In the last class we defined an **ensemble average (EA)** for a flow variable $\varphi(\vec{r},t)$ over many realizations of the same experiment of as

$$\boxed{\bar{\varphi} \equiv \lim_{N\to\infty} \frac{\varphi_1 + \ldots + \varphi_N}{N}}, \qquad \textbf{Definition of EA} \qquad (16)$$

and defined the fluctuating (turbulent) part of $\varphi(\vec{r},t)$ as

$$\boxed{\varphi' \equiv \varphi - \bar{\varphi}} \qquad (18)$$

Or equivalently, $\boxed{\varphi = \bar{\varphi} + \varphi'}$ (17).

To study the structure of the PBL, we will consider the **mean and fluctuating components of our governing equations**. But first we need to understand **properties of ensemble averaging**. Don't worry about the names of the Properties, but make sure you understand what the properties are (boxed results). Also, there may be many different ways to prove these. I'll just show one way.

Property 1: $\boxed{\bar{a} = a}$, where *a* is a "constant" in the sense that it is the same for all realizations (experiments repeated with same conditions). So *a* can be a true constant or it can vary with $\vec{r}$ and *t*, as long as it doesn't change between realizations.

Proof: Use definition of EA with every appearance of φ replaced by a :

$$\bar{a} = \lim_{N\to\infty} \frac{a_1 + \ldots + a_N}{N} = \lim_{N\to\infty} \frac{Na}{N} = a$$

a doesn't change between realizations, so $a_1 = a$, $a_N = a$, etc

Property 1.5: $\boxed{\bar{\bar{\varphi}} = \bar{\varphi}}$. This is a special case of Property 1. Since $\bar{\varphi}$ is constructed as an average over all realizations, once you have it, it's just some number. This number is a "constant" in the sense of Property 1. Property 1 with $a \equiv \bar{\varphi}$ gives $\bar{\bar{\varphi}} = \bar{\varphi}$.

Property 2: $\boxed{\overline{a\,\varphi} = a\,\overline{\varphi}}$, where *a* is a constant (in sense described in Property 1).

Proof: Use definition of EA (with φ replaced by $a\varphi$):

$$\overline{a\varphi} = \underbrace{\lim_{N\to\infty} \frac{a_1\varphi_1 + \ ... \ + \ a_N\varphi_N}{N}}_{\text{use } a_1 = a,\ a_2 = a,\ \text{etc}} = \underbrace{a\left(\lim_{N\to\infty} \frac{\varphi_1 + \ ... \ + \varphi_N}{N}\right)}_{\text{use definition of EA}} = a\bar{\varphi}.$$

Variant of Property 2: $\boxed{\overline{\varphi\, a} = a\,\bar{\varphi}}$.

Property 3 (addition): $\boxed{\overline{\varphi + \psi} = \overline{\varphi} + \overline{\psi}}$, where φ and ψ are **any** two quantities

Proof: Use definition of EA (with φ replaced by $\varphi + \psi$):

$$\overline{\varphi + \psi} = \lim_{N\to\infty} \frac{\varphi_1 + \psi_1 + \ ... \ + \varphi_N + \psi_N}{N} \qquad \text{separate } \varphi \text{ stuff from } \psi \text{ stuff}$$

$$= \underbrace{\lim_{N\to\infty} \frac{\varphi_1 + \ ... \ + \varphi_N}{N}}_{\text{use definition of EA}} + \underbrace{\lim_{N\to\infty}\left(\frac{\psi_1 + \ ... \ + \psi_N}{N}\right)}_{\text{use definition of EA [with } \varphi \text{ replaced by } \psi]}$$

$$= \overline{\varphi} + \overline{\psi}$$

Generalized addition property. Property 3 can be generalized to **any** number of quantitites. For three quantities φ, ψ, χ it becomes: $\boxed{\overline{\varphi + \psi + \chi} = \overline{\varphi} + \overline{\psi} + \bar{\chi}}$.

Proof (for the 3 variable case):

$$\overline{\varphi+\psi+\chi} = \overline{\varphi+(\psi+\chi)} \text{ Use Property 3 with the two variables being } \varphi \text{ and } (\psi+\chi)$$

$$= \overline{\varphi} + \overline{\psi+\chi} \qquad \text{Use Property 3 on the last term}$$

$$= \overline{\varphi} + \overline{\psi} + \bar{\chi}$$

Property 3.5: $\boxed{\overline{\varphi-\psi} = \overline{\varphi} - \overline{\psi}}$, where φ and ψ are **any** two quantities.

Proof: Nearly identical to proof of Property 3. Use definition of EA with every appearance of φ replaced by $\varphi - \psi$ (try it).

Property 4: The ensemble average of the fluctuations is zero: $\boxed{\overline{\varphi'} = 0}$

Proof: Take the ensemble average of $\varphi' \equiv \varphi - \overline{\varphi}$

$$\overline{\varphi'} = \overline{\varphi - \bar{\varphi}} \qquad \text{use subtraction property (3.5)}$$

$$= \bar{\varphi} - \bar{\bar{\varphi}} \qquad \text{use } \bar{\bar{\varphi}} = \bar{\varphi} \text{ (property 1.5)}$$

$$= \bar{\varphi} - \bar{\varphi}$$

$$= 0$$

Property 5: $\boxed{\overline{\varphi'\bar{\psi}} = 0}$. Ensemble average of the **product** of a perturbation with an ensemble average is zero.

Proof: $\bar{\psi}$ is a "constant" in the sense of property 1, so pull it out of the ensemble average (property 2) and get

$$\overline{\varphi'\bar{\psi}} = \bar{\psi}\,\overline{\varphi'} \qquad \text{now use } \overline{\varphi'} = 0 \text{ (property 4)}$$

$$= \bar{\psi}\, 0 = 0$$

Variants of property 5: $\boxed{\overline{\bar{\psi}\varphi'}=0}$, $\boxed{\overline{\bar{\varphi}\psi'}=0}$

Property 6: $\boxed{\overline{\frac{\partial\varphi}{\partial t}}=\frac{\partial\bar{\varphi}}{\partial t}}$

Proof: Replace φ by $\frac{\partial\varphi}{\partial t}$ in the definition of EA:

$$\overline{\frac{\partial\varphi}{\partial t}}=\lim_{N\to\infty}\frac{\left(\frac{\partial\varphi}{\partial t}\right)_1+\ldots+\left(\frac{\partial\varphi}{\partial t}\right)_N}{N}=\lim_{N\to\infty}\frac{\frac{\partial}{\partial t}(\varphi_1+\ldots+\varphi_N)}{N}$$

$$=\frac{\partial}{\partial t}\left(\lim_{N\to\infty}\frac{\varphi_1+\ldots+\varphi_N}{N}\right)=\frac{\partial\bar{\varphi}}{\partial t}$$

Property 7: $\boxed{\overline{\frac{\partial\varphi}{\partial x}}=\frac{\partial\bar{\varphi}}{\partial x}}$, $\boxed{\overline{\frac{\partial\varphi}{\partial y}}=\frac{\partial\bar{\varphi}}{\partial y}}$, $\boxed{\overline{\frac{\partial\varphi}{\partial z}}=\frac{\partial\bar{\varphi}}{\partial z}}$

Proof: Proofs are virtually identical to that of Property 6. Replace *t* with *x*, *y* or *z*.

Property 8: $\boxed{\overline{\varphi\,\psi}=\bar{\varphi}\,\bar{\psi}+\overline{\varphi'\psi'}}$, where φ and ψ are **any** two quantities.

Proof: In $\overline{\varphi\,\psi}$, decompose φ and ψ into their mean and fluctuating parts:

$\overline{\varphi\,\psi}=\overline{(\bar{\varphi}+\varphi')(\bar{\psi}+\psi')}$	Expand it out
$=\overline{\bar{\varphi}\bar{\psi}+\bar{\varphi}\psi'+\varphi'\bar{\psi}+\varphi'\psi'}$	Use generalized addition property
$=\overline{\bar{\varphi}\bar{\psi}}+\overline{\bar{\varphi}\psi'}+\overline{\varphi'\bar{\psi}}+\overline{\varphi'\psi'}$	Apply property 5 to 2nd and 3rd terms
$=\overline{\bar{\varphi}\bar{\psi}}+\overline{\varphi'\psi'}$	Use property 2 and 1.5 on first term
$=\bar{\varphi}\bar{\psi}+\overline{\varphi'\psi'}$	

Reynolds Averaging

In real life, you can't run an infinite number of experiments ($N \to \infty$) to get an ensemble average. Also, it's unlikely you'll have the financial, computational or time resources to conduct even a large number of experiments ($N \to$ big but not ∞).

Reynolds averaging allows us to **approximate** the ensemble average by other kinds of averages, depending on the statistical properties of the flow, such as

Stationarity (stationary flow**)**: statistical properties of flow do not depend on time.

Homogeneity: statistical properties of flow do not vary in space.

Horizontal homogeneity: statistical properties of flow do not vary *x* or *y*.

If flow is **stationary** then we can perform a **Reynolds-average in time** by replacing the ensemble average with a time average (theoretically it should be over an infinite time interval *T*, but in practice *T* has to be a "large" but not infinite interval):

$$\overline{\varphi}^{t}(x,y,z) \equiv \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \varphi(x,y,z,t)\,dt\,,$$

where t_0 is an arbitrary start time.

If flow is **horizontally homogeneous** then we can perform a **Reynolds-average in space (*x* and *y*)** by replacing the ensemble average with a horizontal average (theoretically it should be over infinite intervals *L* in *x* and *y*, but in practice *L* has to be finite):

$$\overline{\varphi}^{xy}(z,t) \equiv \lim_{L\to\infty} \frac{1}{L^2} \int_{x_0}^{x_0+L} \int_{y_0}^{y_0+L} \varphi(x,y,z,t)\, dx\, dy\,,$$

where x_0 and y_0 define an arbitrary starting location.

Important! The properties of ensemble-averaging are assumed to hold true for Reynolds averaging!