LECTURE 32
The Planetary Boundary Layer (continued)

Flux form of the Boussinesq equations of motion

The Boussinesq equations of motion (lecture 30) are rewritten slightly as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_c}\frac{\partial \boxed{p}}{\partial x}+f\,v+\nu\nabla^2 u\,,$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho_c}\frac{\partial \boxed{p}}{\partial y}-f\,u+\nu\nabla^2 v\,,$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_c}\frac{\partial \boxed{p}}{\partial z}+\frac{\boxed{\theta}}{\theta_c}g+\nu\nabla^2 w\,,$$

where ***p* is the perturbation pressure** (formerly written as p') and $\boldsymbol{\theta}$ **is the perturbation potential temperature** (formerly written as θ').

Prior to averaging, we put the advection terms in "flux form" (I'll explain the terminology). To do that, use the product rule $d(AB)=A\,dB+B\,dA$ in the form,

$$A\,dB=d(AB)-B\,dA\,,$$

where *d* is shorthand for $\partial/\partial x$, $\partial/\partial y$, or $\partial/\partial z$. So, the advection terms in the *x*-comp equation of motion become:

$$\underset{A=u\,,\,B=u}{u\frac{\partial u}{\partial x}}+\underset{A=v\,,\,B=u}{v\frac{\partial u}{\partial y}}+\underset{A=w\,,\,B=u}{w\frac{\partial u}{\partial z}}=\left(\frac{\partial uu}{\partial x}-u\frac{\partial u}{\partial x}\right)+\left(\frac{\partial uv}{\partial y}-u\frac{\partial v}{\partial y}\right)+\left(\frac{\partial uw}{\partial z}-u\frac{\partial w}{\partial z}\right)$$

$$=\frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}-u\boxed{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}$$

Last term is 0 (from incompressibility condition), so we get,

$$\boxed{u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} \;=\; \frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}}.$$

Similarly, the advection terms in the *y* and *z* component equations of motion become:

$$\boxed{u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} \;=\; \frac{\partial uv}{\partial x}+\frac{\partial vv}{\partial y}+\frac{\partial vw}{\partial z}},$$

$$\boxed{u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} \;=\; \frac{\partial uw}{\partial x}+\frac{\partial vw}{\partial y}+\frac{\partial ww}{\partial z}}.$$

We thus get the Boussinesq equations of motion in **flux form** as:

$$\frac{\partial u}{\partial t}+\frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}=-\frac{1}{\rho_c}\frac{\partial \boxed{p}}{\partial x}+f\,v+\nu\nabla^2 u\,, \tag{19}$$

$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial vv}{\partial y}+\frac{\partial vw}{\partial z}=-\frac{1}{\rho_c}\frac{\partial \boxed{p}}{\partial y}-f\,u+\nu\nabla^2 v, \tag{20}$$

$$\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial vw}{\partial y}+\frac{\partial ww}{\partial z}=-\frac{1}{\rho_c}\frac{\partial \boxed{p}}{\partial z}+\frac{\boxed{\theta}}{\theta_c}g+\nu\nabla^2 w\,, \tag{21}$$

Ensemble-average (Reynolds-average) of incompressibility condition

Take EA of the incompressibility condition, get:

$$\overline{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}}=0 \qquad \text{use addition property 3}$$

$$\overline{\frac{\partial u}{\partial x}}+\overline{\frac{\partial v}{\partial y}}+\overline{\frac{\partial w}{\partial z}}=0$$

Use fact that EA operator $\overline{}$ commutes with spatial derivative operators (Property 7)

$$\boxed{\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}+\frac{\partial \overline{w}}{\partial z}=0} \tag{22}$$

Ensemble-average (Reynolds-average) of the equations of motion

Take ensemble average of the flux-form of the x-component equation of motion:

$$\overline{\frac{\partial u}{\partial t}+\frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}}=\overline{-\frac{1}{\rho_C}\frac{\partial p}{\partial x}+f\,v+\nu\nabla^2 u} \quad \text{use addition property 3}$$

$$\overline{\frac{\partial u}{\partial t}}+\overline{\frac{\partial uu}{\partial x}}+\overline{\frac{\partial uv}{\partial y}}+\overline{\frac{\partial uw}{\partial z}}=\overline{-\frac{1}{\rho_C}\frac{\partial p}{\partial x}}+\overline{f\,v}+\overline{\nu\nabla^2 u}$$

Use fact that EA operator $\overline{}$ commutes with spatial and time derivatives (Properties 6 and 7), and constants pass through EA operator (Property 1), get

$$\frac{\partial \bar{u}}{\partial t}+\frac{\partial \overline{uu}}{\partial x}+\frac{\partial \overline{uv}}{\partial y}+\frac{\partial \overline{uw}}{\partial z}=-\frac{1}{\rho_C}\frac{\partial \bar{p}}{\partial x}+f\,\bar{v}+\nu\nabla^2\bar{u}$$

Apply Property 8 (for products), $\overline{\varphi\,\psi}\;=\;\overline{\varphi}\,\overline{\psi}+\overline{\varphi'\,\psi'}$, to the nonlinear terms:

$\overline{u\,u}\;=\;\bar{u}\,\bar{u}+\overline{u'\,u'}$ — $\overline{u'\,u'}$ is the **turbulent variance of *u***

$\overline{u\,v}\;=\;\bar{u}\,\bar{v}+\overline{u'\,v'}$ — $\overline{u'\,v'}$ is the **turbulent co-variance of *u* and *v***

$\overline{u\,w}\;=\;\bar{u}\,\bar{w}+\overline{u'\,w'}$ — $\overline{u'\,w'}$ is the **turbulent co-variance of *u* and *w***

So we get:

$$\frac{\partial \bar{u}}{\partial t}+\frac{\partial}{\partial x}\Big(\bar{u}\,\bar{u}+\overline{u'\,u'}\Big)+\frac{\partial}{\partial y}\Big(\bar{u}\,\bar{v}+\overline{u'\,v'}\Big)+\frac{\partial}{\partial z}\Big(\bar{u}\,\bar{w}+\overline{u'\,w'}\Big)=-\frac{1}{\rho_C}\frac{\partial \bar{p}}{\partial x}+f\,\bar{v}+\nu\nabla^2\bar{u}$$

Take all turbulent variance and co-variance terms over to the right hand side.

$$\frac{\partial \bar{u}}{\partial t}+\frac{\partial \bar{u}\,\bar{u}}{\partial x}+\frac{\partial \bar{u}\,\bar{v}}{\partial y}+\frac{\partial \bar{u}\,\bar{w}}{\partial z}=-\frac{1}{\rho_C}\frac{\partial \bar{p}}{\partial x}+f\,\bar{v}-\frac{\partial \overline{u'u'}}{\partial x}-\frac{\partial \overline{u'v'}}{\partial y}-\frac{\partial \overline{u'w'}}{\partial z}+\nu\nabla^2\bar{u}$$

Expand out the advection terms to get them out of flux form:

$$\frac{\partial \bar{u}\,\bar{u}}{\partial x}+\frac{\partial \bar{u}\,\bar{v}}{\partial y}+\frac{\partial \bar{u}\,\bar{w}}{\partial z}=\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{u}\frac{\partial \bar{v}}{\partial y}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{u}\frac{\partial \bar{w}}{\partial z}+\bar{w}\frac{\partial \bar{u}}{\partial z}$$

$$=\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}+\bar{u}\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}\right)^{\rightarrow 0 \text{ from (22)}}$$

$$=\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z} \quad \text{now they look like usual advection terms}$$

So, the **ensemble-averaged x-comp equation of motion becomes**

$$\boxed{\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}=-\frac{1}{\rho_C}\frac{\partial \bar{p}}{\partial x}+f\,\bar{v}-\frac{\partial \overline{u'u'}}{\partial x}-\frac{\partial \overline{u'v'}}{\partial y}-\frac{\partial \overline{u'w'}}{\partial z}+\nu\nabla^2\bar{u}} \qquad (23)$$

Similarly, we obtain the **ensemble-averaged y and z-comp equations of motion as**

$$\boxed{\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}=-\frac{1}{\rho_C}\frac{\partial \bar{p}}{\partial y}-f\,\bar{u}-\frac{\partial \overline{u'v'}}{\partial x}-\frac{\partial \overline{v'v'}}{\partial y}-\frac{\partial \overline{v'w'}}{\partial z}+\nu\nabla^2\bar{v}} \qquad (24)$$

$$\boxed{\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}=-\frac{1}{\rho_C}\frac{\partial \bar{p}}{\partial z}+\frac{\bar{\theta}}{\theta_c}g-\frac{\partial \overline{u'w'}}{\partial x}-\frac{\partial \overline{v'w'}}{\partial y}-\frac{\partial \overline{w'w'}}{\partial z}+\nu\nabla^2\bar{w}} \qquad (25)$$

These are prognostic equations for the mean velocity components $\bar{u}$, $\bar{v}$, and $\bar{w}$. Each of them "looks" like the equation of motion that gave rise to it, except for the presence of new forcing terms: the turbulent variances/co-variances.