LECTURE 34
The Planetary Boundary Layer (continued)

Ekman layer (continued)

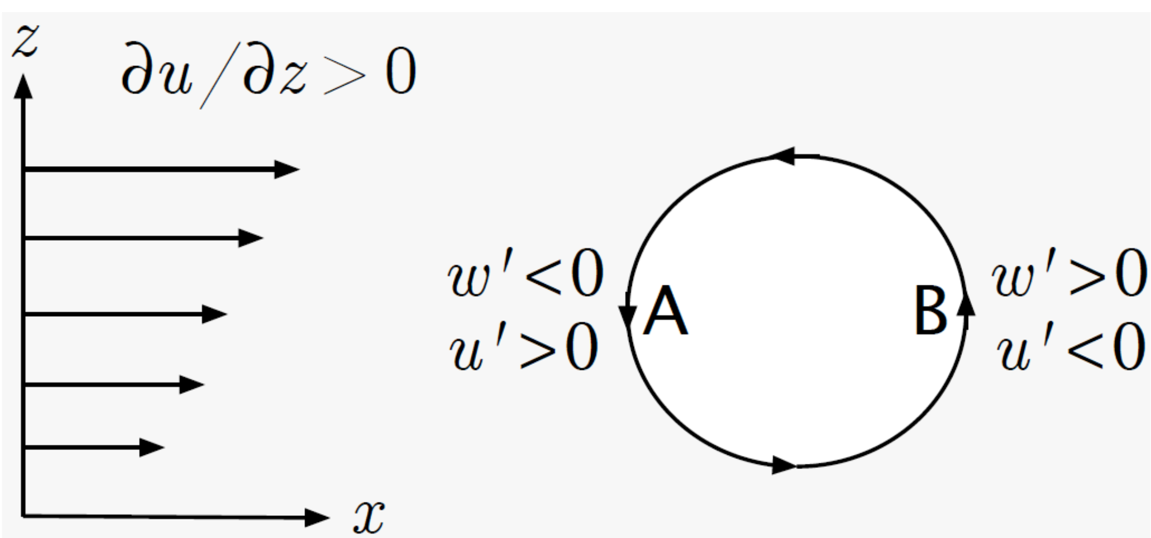
We close the system using an **eddy viscosity parameterization** (also due to Boussinesq) in which the Reynolds stresses $\overline{u'w'}$ and $\overline{v'w'}$ are set proportional to vertical gradients of the corresponding mean velocity components:

$$\overline{u'w'} = -K\frac{d\overline{u}}{dz}, \tag{33}$$

$$\overline{v'w'} = -K\frac{d\overline{v}}{dz}. \tag{34}$$

Here K is the eddy viscosity coefficient (or just eddy viscosity). **Relating Reynolds stresses to velocity gradients through an eddy viscosity coefficient** is analogous to **relating viscous shear stresses to velocity gradients through a kinematic viscosity coefficient** (covered in Dynamics I).

Does this parameterization make sense? Consider an eddy in a shear flow:



In rising branch (point B, where $w' > 0$) statistically slower air is transported upward (since $d\overline{u}/dz > 0$, air beneath B has a smaller value of u than air at B). So u value transported past B is less than mean u at B, so $\boldsymbol{u' < 0}$. **The stronger the shear, the larger this deficit is, and the larger the magnitude of** $\boldsymbol{u'}$ **is**. So $u'w' < 0$, with a magnitude that increases with increasing $d\overline{u}/dz$. This suggests $u'w'$ is proportional to $-d\overline{u}/dz$ (minus sign since $d\overline{u}/dz$ is positive while $u'w'$ is negative).

Similar reasoning in descending branch (point A, $w' < 0$), where faster air is transported downward ($u' > 0$) still yields $u'w' < 0$, with a magnitude that increases with increasing $d\overline{u}/dz$. Again, this suggests $u'w'$ is proportional to $-d\overline{u}/dz$.

Get same result in clockwise-spinning eddies (it's counter-clockwise in diagram).

So, for an ensemble average, we expect $\overline{u'w'} = -K\dfrac{d\overline{u}}{dz}$, where K is a proportionality factor that we call the eddy viscosity.

Eddy viscosity parameterization (closure) is commonly used in mesoscale and climate models – with K specified in different ways, e.g., related to wind shear and static stability, or to a predicted turbulent kinetic energy and a length scale. Here, we will simply take K to be constant (as did Ekman).

Applying (33) and (34) in (31) and (32) yields

$$0 = f\left(\overline{v} - v_g\right) + \left(K + \nu\right)\frac{d^2\overline{u}}{dz^2},$$

$$0 = -f\left(\overline{u} - u_g\right) + \left(K + \nu\right)\frac{d^2\overline{v}}{dz^2},$$

Daytime K ranges from 10 to 200 $\mathrm{m^2\,s^{-1}}$, while ν is ~$1.5\times10^{-5}\,\mathrm{m^2\,s^{-1}}$ (~ million to ~10 million times smaller than K). So we can safely neglect ν and just work with

$$0 = f\left(\overline{v} - v_g\right) + K\frac{d^2\overline{u}}{dz^2}, \tag{35}$$

$$0 = -f\left(\overline{u} - u_g\right) + K\frac{d^2\overline{v}}{dz^2}. \tag{36}$$

Decompose $\overline{u}$ into a geostrophic component u_g and a part that's not geostrophic (ageostrophic wind component u_a): $\overline{u} = u_g + \overline{u}_a$. Same for $\overline{v}$: $\overline{v} = v_g + \overline{v}_a$. In other words, we define ageostrophic wind components by:

$$\overline{u}_a \equiv \overline{u} - u_g, \tag{37}$$

$$\overline{v}_a \equiv \overline{v} - v_g. \tag{38}$$

Since the geostrophic wind in this problem is independent of *z* (we showed the horizontal perturbation pgf is independent of *z*), we can rewrite (35) and (36) completely in terms of ageostrophic wind components:

$$0 = f\overline{v}_a + K\frac{d^2\overline{u}_a}{dz^2}, \tag{39}$$

$$0 = -f\overline{u}_a + K\frac{d^2\overline{v}_a}{dz^2}. \tag{40}$$

There are (at least) two different ways to solve the coupled ODEs (39) and (40):

(i) **Standard way**. "Uncouple" the equations by eliminating one variable in favor of another. For example, write $\overline{v}_a$ in terms of $\overline{u}_a$. From (39): $\overline{v}_a = -(K/f)\, d^2\overline{u}_a/dz^2$. Plugging this into (40) yields a 4th order ODE for just one variable:

$$\frac{d^4\overline{u}_a}{dz^4} + \frac{f^2}{K^2}\overline{u}_a = 0\,.$$

Eliminating $\overline{u}_a$ in favor of $\overline{v}_a$ would lead to an analogous 4th order ODE for $\overline{v}_a$.

(ii) **Linear combination "trick"**. Combine the 2 ODEs into 1 ODE in a single **new** dependent variable, a linear combination of the original variables. If this trick works (it often doesn't), it really simplifies things. **We'll solve (39) and (40) this trick way**.

Multiply (40) by *i* and add the resulting equation to (39):

$$0 = f(\bar{v}_a - i\bar{u}_a) + K\frac{d^2}{dz^2}(\bar{u}_a + i\bar{v}_a), \qquad (41)$$

Define a new dependent variable "*M*",

$$M \equiv \bar{u}_a + i\bar{v}_a\,. \qquad (42)$$

This simplifies last term in (41), but what about the first term? Write $\bar{u}_a$ or $\bar{v}_a$ in terms of *M* (doesn't matter which one; get same result in the end). Okay, I'll use $\bar{u}_a$: $\bar{u}_a = M - i\bar{v}_a$. Plug that into first term:

$$f(\bar{v}_a - i\bar{u}_a) = f\left[\bar{v}_a - i(M - i\bar{v}_a)\right] = f\left[\bar{v}_a - iM + i^2\bar{v}_a\right] \qquad \text{use } i^2 = -1$$

$$= f\left[\bar{v}_a - iM - \bar{v}_a\right] = -ifM$$

So first term also simplifies! So (41) reduces to:

$$0 = -ifM + K\frac{d^2M}{dz^2}\,. \qquad (43)$$

It's a 2nd order linear constant coefficient homogeneous ODE. Seek solutions in the form of exponentials. Plug the trial solution $M \sim e^{qz}$ into (43), get:

$$0 = -ife^{qz} + K\frac{d^2e^{qz}}{dz^2}$$

$$0 = -ife^{qz} + Kq^2e^{qz} \qquad \div \text{ by } e^{qz}$$

$$0 = -if + Kq^2$$

$$q^2 = i\frac{f}{K}$$ Take square root

$$q = \pm\sqrt{i}\sqrt{\frac{f}{K}}$$ We really get two possible roots (plus and minus)

Use fact that $\sqrt{i} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ (you'll prove it in the problem set).

$$q = \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\sqrt{\frac{f}{K}}$$

Simplifies a bit:

$$q = \pm\left(1+i\right)\sqrt{\frac{f}{2K}}$$

So we get two *q* as:

$$q_1 = \left(1+i\right)\sqrt{\frac{f}{2K}}, \qquad q_2 = -\left(1+i\right)\sqrt{\frac{f}{2K}}. \tag{44}$$

Note: We'll restrict attention to the **Northern hemisphere**. **So $f > 0$**. And *K* is some positive constant. So $\sqrt{\frac{f}{2K}}$ is real.