LECTURE 35
The Planetary Boundary Layer (continued)

Ekman layer (continued)

Recap: we're solving the *x*- and *y*-component Reynolds-averaged equations of motion using an eddy viscosity closure for the Reynolds stresses. We combined two coupled ODEs for $\bar{u}_a \equiv \bar{u} - u_g$, and $\bar{v}_a \equiv \bar{v} - v_g$ into a single ODE for $M \equiv \bar{u}_a + i\bar{v}_a$,

$$0 = -ifM + K\frac{d^2M}{dz^2}. \tag{43}$$

This equation has solutions of the form $M \sim e^{qz}$ for two q:

$$q_1 = (1+i)\sqrt{\frac{f}{2K}}, \qquad q_2 = -(1+i)\sqrt{\frac{f}{2K}}. \tag{44}$$

In terms of the parameter

$$\gamma \equiv \sqrt{\frac{f}{2K}}, \quad \text{which is real (in N. hemisphere) and positive,}$$

these q's become: $q_1 = \gamma(1+i)$, $q_2 = -\gamma(1+i)$. The general solution of (43) is a linear combination of the solutions associated with these q:

$$M(z) \;=\; C\exp[\gamma(1+i)\,z] + D\exp[-\gamma(1+i)z], \tag{45}$$

where C and D are constants. Rewrite the first exponential:

$$\exp[\gamma(1+i)\,z] = \exp(\gamma z + i\gamma z) = e^{\gamma z}\,e^{i\gamma z} \qquad \text{use Euler's formula}$$

$$= e^{\gamma z}\left(\cos\gamma z + i\sin\gamma z\right)$$

Similarly, for the second exponential, we have

$$\exp\left[-\gamma(1+i)\,z\right] = \exp(-\gamma z - i\gamma z) = e^{-\gamma z}\,e^{-i\gamma z}$$

$$= e^{-\gamma z}\left(\cos\gamma z - i\sin\gamma z\right)$$

Put 'em back into (45), get:

$$M(z) = C\,e^{\gamma z}\left(\cos\gamma z + i\sin\gamma z\right) + D\,e^{-\gamma z}\left(\cos\gamma z - i\sin\gamma z\right) \quad (46)$$

Now consider boundary conditions. Far above the surface ($z \to \infty$), the wind becomes geostrophic: $\bar{u} \to u_g$ and $\bar{v} \to v_g$. So $\bar{u}_a = \bar{u} - u_g \to u_g - u_g = 0$ and $\bar{v}_a \equiv \bar{v} - v_g \to v_g - v_g = 0$. So we impose:

$$\lim_{z\to\infty} M = \lim_{z\to\infty}\left(\bar{u}_a + i\bar{v}_a\right) = 0, \qquad \text{[upper boundary condition for } M\text{]}$$

Looking at (46) as $z \to \infty$ we see that the term multiplying *D* goes to 0 since $e^{-\gamma z} \to 0$ (γ being positive), but the term multiplying *C* blows up since $e^{\gamma z} \to \infty$, (γ being positive). To ensure $M \to 0$, **we must set *C* = 0**. Then (46) becomes

$$M(z) = D\,e^{-\gamma z}\left(\cos\gamma z - i\sin\gamma z\right). \quad (47)$$

Now consider the surface conditions: no-slip on *u* and *v*; *u*(0) = 0 and *v*(0) = 0. So $\bar{u}_a(0) = \bar{u}(0) - u_g = 0 - u_g = -u_g$ and $\bar{v}_a(0) = \bar{v}(0) - v_g = 0 - v_g = -v_g$. So:

$$M(0) = \bar{u}_a(0) + i\bar{v}_a(0) = -\left(\bar{u}_g + i\bar{v}_g\right). \qquad \text{[lower boundary condition for } M\text{]}$$

So, set *z* = 0 in (47) and use the lower boundary condition for *M*:

$$M(0) = D\,e^{-0}\left[\cos(0) - i\sin(\gamma 0)\right] \qquad \text{plug in lower b.c. for M}$$

$$-\left(\bar{u}_g + i\bar{v}_g\right) = D\left[1 - 0\right] = D \qquad \therefore \boxed{D = -\left(\bar{u}_g + i\bar{v}_g\right)}$$

So now (47) becomes

$$M(z) = -\left(\overline{u}_g + i\overline{v}_g\right)e^{-\gamma z}\left(\cos\gamma z - i\sin\gamma z\right) .$$

Plug in $M \equiv \overline{u}_a + i\overline{v}_a$.

$$\overline{u}_a + i\overline{v}_a = -\left(\overline{u}_g + i\overline{v}_g\right)e^{-\gamma z}\left(\cos\gamma z - i\sin\gamma z\right) \tag{48}$$

To find out what $\overline{u}_a$ and $\overline{v}_a$ are, we need to take real and imaginary parts of (48). So, rewrite the right hand side of (48) to make clear what its real and imaginary parts are:

$$\overline{u}_a + i\overline{v}_a = -e^{-\gamma z}\left[\left(\overline{u}_g + i\overline{v}_g\right)\left(\cos\gamma z - i\sin\gamma z\right)\right] \quad \text{expand out right hand side}$$

$$= -e^{-\gamma z}\left[\overline{u}_g\cos\gamma z - i\overline{u}_g\sin\gamma z + i\overline{v}_g\cos\gamma z - i(i\overline{v}_g)\sin\gamma z\right]$$

In the last term use fact that $-ii = -i^2 = -(-1) = 1$. So we get

$$\overline{u}_a + i\overline{v}_a = -e^{-\gamma z}\left(\overline{u}_g\cos\gamma z + \overline{v}_g\sin\gamma z - i\overline{u}_g\sin\gamma z + i\overline{v}_g\cos\gamma z\right) \tag{49}$$

Now take real and imaginary parts of this equation. The real part gives:

$$\overline{u}_a = -e^{-\gamma z}\left(\overline{u}_g\cos\gamma z + \overline{v}_g\sin\gamma z\right)$$

The imaginary part gives:

$$\overline{v}_a = -e^{-\gamma z}\left(-\overline{u}_g\sin\gamma z + \overline{v}_g\cos\gamma z\right)$$

Lastly, rewrite these in terms of the mean winds $\overline{u} = u_g + \overline{u}_a$, and $\overline{v} = v_g + \overline{v}_a$:

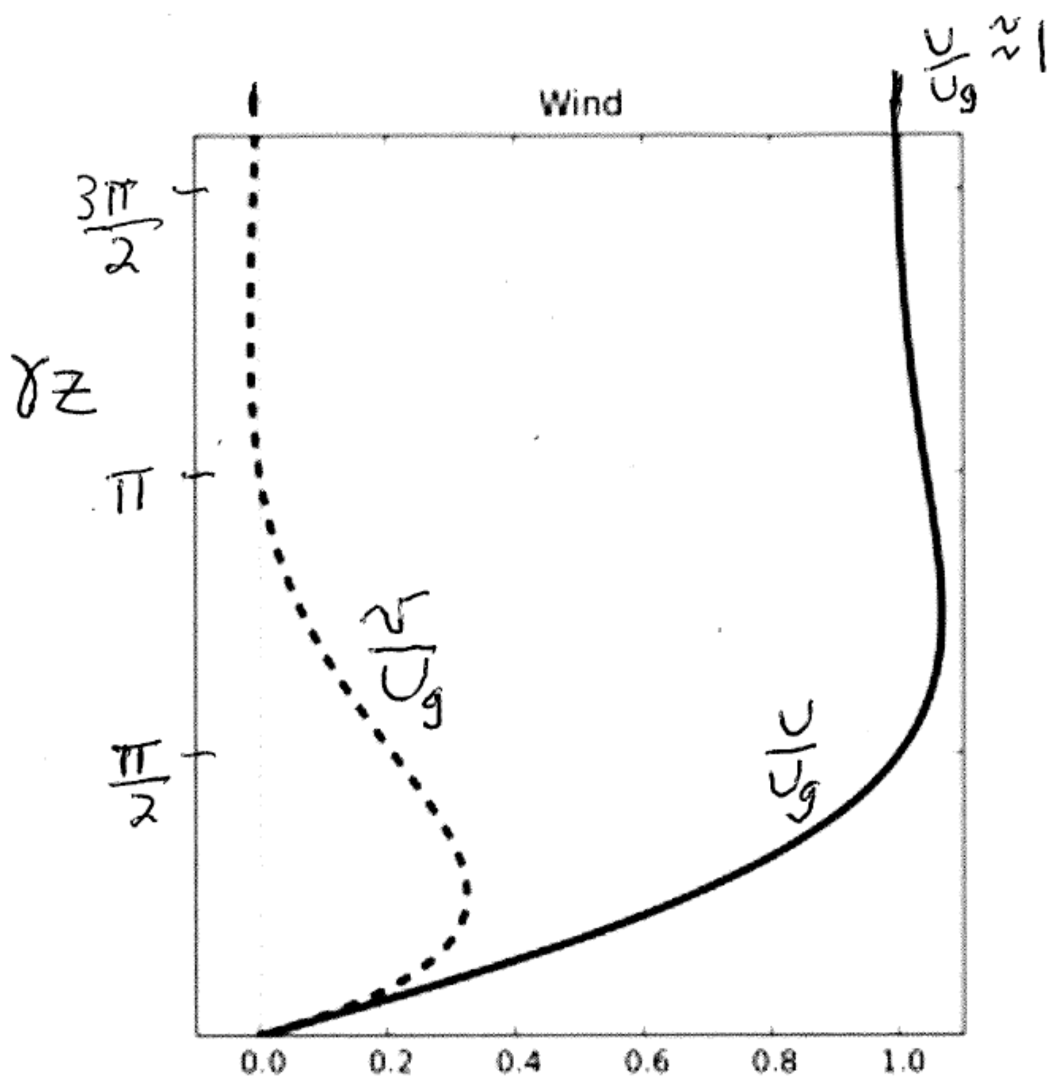
$$\overline{u} = \overline{u}_g\left(1 - e^{-\gamma z}\cos\gamma z\right) - \overline{v}_g e^{-\gamma z}\sin\gamma z, \tag{50}$$

$$\overline{v} = \overline{v}_g\left(1 - e^{-\gamma z}\cos\gamma z\right) + \overline{u}_g e^{-\gamma z}\sin\gamma z, \tag{51}$$

where $\gamma \equiv \sqrt{\dfrac{f}{2K}}$.

[Reality check: Verify using (50) and (51) that flow becomes geostrophic far above surface and goes to zero at surface. Yep, it checks out.]

Consider a new (rotated) coordinate system in which **new *x* axis is aligned with geostrophic wind** (all of geostrophic wind vector is in new *x*-direction, so $v_g = 0$). So ***y* axis points toward low pressure!).** In this new system, (50) and (51) become: $\bar{u} = \bar{u}_g\left(1 - e^{-\gamma z}\cos\gamma z\right)$, $\bar{v} = \bar{u}_g e^{-\gamma z}\sin\gamma z$. These are graphed below:



Non-dimensional wind components (no units)

Within boundary layer there's a **cross-isobar flow toward low pressure (*v* > 0)**.

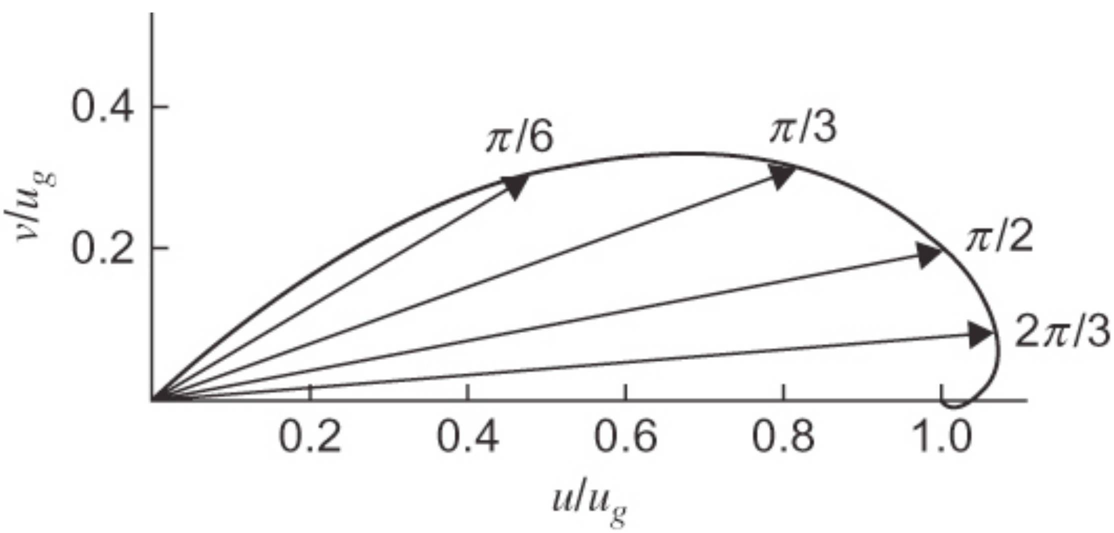
Various ways to define height of boundary layer. One way is to say it's the height H where $\gamma H = \pi$ (so $e^{-\gamma H} = e^{-\pi} \approx 0.05$). At this height u has nearly reached its geostrophic value and v has nearly returned to 0 (see diagram above).

$$H = \frac{\pi}{\gamma} = \frac{\pi}{\sqrt{\frac{f}{2K}}} = \pi\sqrt{\frac{2K}{f}}\,.$$

For typical values $K = 10\,\mathrm{m}^2\,\mathrm{s}^{-1}$, $f = 8\times10^{-5}\,\mathrm{s}^{-1}$, we obtain H as: H ~ 1.5 km.

Ekman spiral

(hodograph of v versus u for γz ranging from ground through top of boundary layer):



Theoretical Ekman spiral (top dashed curve) versus a real Ekman spiral (solid curve):