LECTURE 38
Waves in the Atmosphere

Wave parameters (for any kind of wave)

We'll work with waves of the form:

$$\text{some flow property} \sim A \sin\left[\frac{2\pi}{\lambda}\left(x-ct\right)\right] \quad \text{or} \quad A \cos\left[\frac{2\pi}{\lambda}\left(x-ct\right)\right],$$

where

A is **amplitude**

$\frac{2\pi}{\lambda}\left(x-ct\right)$ is **phase** of the wave.

λ is **wavelength**,

c is **phase speed**.

Why consider waves of this form?

- Many waves "look" like sines or cosines.

- Many waves are governed by linear constant coefficient ODEs and PDEs. These have sine and cosine solutions. Can superimpose these solutions via Fourier analysis to get solutions involving arbitrary initial conditions or arbitrary boundary forcings.

A closer look at the wave parameters:

phase: $\frac{2\pi}{\lambda}\left(x-ct\right)$

Wave repeats itself when **phase changes by 2π**. So, at a fixed moment in time, phase changes by 2π when x changes by λ. Hence the name wavelength for λ.

$k \equiv \frac{2\pi}{\lambda}$ is **wavenumber**. It's the number of oscillations in a (dimensional) length of 2π. For example, if $\lambda = 1$ (meters) then over a distance of 2π meters we see about 6 oscillations: $k \equiv \frac{2\pi}{1\,\mathrm{m}} \cong \boxed{6.28}\,\mathrm{m}^{-1}$.

Think: long waves --> small k

short waves --> big k

Look at phase again, and but now consider a fixed x. Phase changes by 2π when time changes by $\frac{\lambda}{c}$. So, wave **period** is: $T \equiv \frac{\lambda}{c}$.

$\nu \equiv \frac{1}{T}$ is **frequency**. Number of oscillations per unit time.

$\omega \equiv \frac{2\pi}{T} = 2\pi\nu$ is **circular (or radian) frequency**. It's the number of oscillations in a (dimensional) 2π length of time. For example, if T = 1 (second) then over a duration of 2π seconds, there are about 6 oscillations: $\omega = \frac{2\pi}{1\mathrm{s}} \cong \boxed{6.28}\,\mathrm{s}^{-1}$

Since $T \equiv \frac{\lambda}{c} \quad \rightarrow \quad \omega = \frac{2\pi}{T} = 2\pi\frac{c}{\lambda} = ck$

$$\therefore \quad \boxed{c = \frac{\omega}{k}}$$

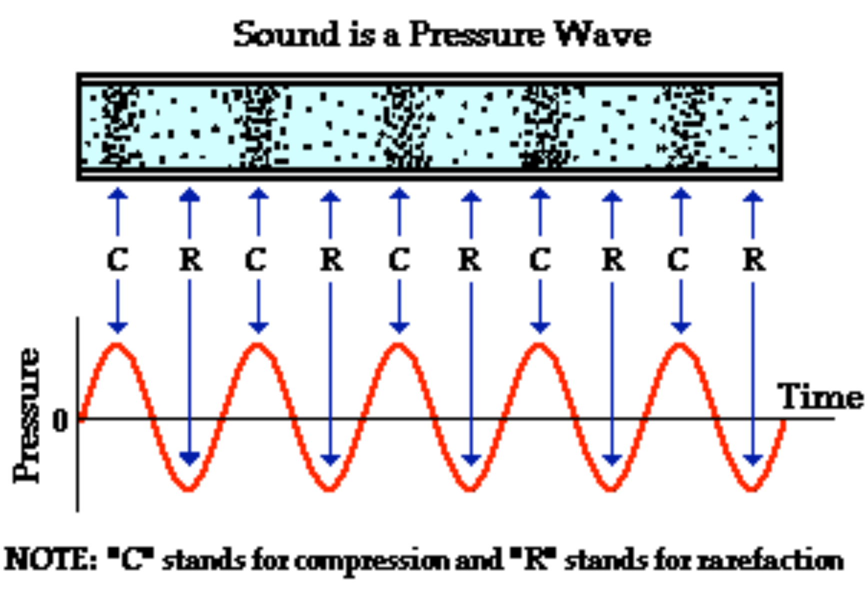
So we're led to the equivalent expressions:

$$\sin\left[\frac{2\pi}{\lambda}\left(x - ct\right)\right] \quad \text{or} \quad \sin\left[k\left(x - ct\right)\right] \quad \text{or} \quad \sin\left(kx - \omega t\right)$$

Acoustic (sound) waves

Sound is a **pressure wave**. It propagates via alternating adiabatic expansions (rarefactions) and compressions. The **compression zone is a high *p* zone**. The **rarefaction zone is a low *p* zone**. Sound waves are created through mechanical vibrations (e.g., vibrating engines, shedding of vortices from branches, turbulent motion, crying brats) and then propagate away from source. Air **velocities** in sound waves are **parallel (and anti-parallel) to direction of propagation**, so these are longitudinal waves, not transverse (shear) waves.

As in most waves, **speed of propagation and speed of parcels are two different things**! A sound wave might propagate in 1 direction at 350 m/s, while the velocity of the air parcels is "back and forth" at a few mm/s. [Similarly, in a human wave at a football game, pattern moves laterally at > 10 m/s but people don't move laterally at all; and the vertical speed of arms is much less than 10 m/s].



Sound waves are often created with spherical symmetry (point source), but as the further they propagate away from the source, the more the phase fronts look planar so we can study them as 1D structures.

Eqns of motion, energy and mass conservation for 1D sound waves

Consider simplest case that supports sound waves: 1D flow. For example, let *u* and all thermodynamic variables vary only with *x* and *t*, while *v* = *w* = 0 everywhere, for all *t*. Viscous and Coriolis terms aren't important to this phenomenon, so neglect them. In this derivation, whenever it's convenient to write a thermodynamic variable as a natural log (e.g., $\ln\theta$), do it!

x-component eqn of motion:

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{1}$$

Mass conservation eqn. Use an exact one (NOT incompressibility condition): $\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{u} = 0$. Using $\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{D\ln\rho}{Dt}$ and 1D assumption, this becomes:

$$\frac{D\ln\rho}{Dt} + \frac{\partial u}{\partial x} = 0. \tag{2}$$

Thermodynamic energy eqn for adiabatic motion: $D\theta/Dt = 0$, which also means:

$$\frac{D\ln\theta}{Dt} = 0. \tag{3}$$

Here $$\theta \equiv T\left(\frac{1000\,\text{mb}}{p}\right)^{R/c_p}. \tag{4}$$

We have 5 variables *u*, *p*, *T*, θ, ρ in 4 unknowns. Need 1 more equation to close the system. Bring in the **ideal gas law**,

$$p = \rho R T. \tag{5}$$

Now the system is closed.

Lets eliminate $\ln\theta$ from the equation set. First take ln of (4), get

$$\ln\theta = \ln T - \frac{R}{c_p}\ln p + \frac{R}{c_p}\ln\left(1000\text{mb}\right). \tag{6}$$

Applying this in (3) yields

$$0 = \frac{D\ln T}{Dt} - \frac{R}{c_p}\frac{D\ln p}{Dt}. \tag{7}$$

So now $\ln\theta$ is gone. Now lets get rid of ln*T*. Take ln of ideal gas law (5) and rearrange it to get an expression for ln*T*:

$$\ln T = \ln p - \ln\rho - \ln R. \tag{8}$$

Plugging this into (7) removes ln*T* from the problem:

$$0 = \frac{D\ln p}{Dt} - \frac{D\ln\rho}{Dt} - \frac{R}{c_p}\frac{D\ln p}{Dt} \qquad \text{combine the ln}p\text{ terms}$$

$$0 = \left(1 - \frac{R}{c_p}\right)\frac{D\ln p}{Dt} - \frac{D\ln\rho}{Dt} \qquad \text{rewrite } 1 - R/c_p \text{ (below)}$$

$$0 = \frac{1}{\gamma}\frac{D\ln p}{Dt} - \frac{D\ln\rho}{Dt} \tag{9}$$

where $\gamma \equiv \frac{c_p}{c_v} \cong 1.4$.

__________________________Scratch paper for $1 - \frac{R}{c_p}$

Since $R = c_p - c_v$:

$$1 - \frac{R}{c_p} = 1 - \frac{(c_p - c_v)}{c_p} = 1 - 1 + \frac{c_v}{c_p} = \frac{c_v}{c_p}$$

Define $\gamma \equiv \frac{c_p}{c_v}$. $\qquad \therefore\ 1 - \frac{R}{c_p} = \frac{1}{\gamma}$

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Now lets go after $\ln \rho$. Get $D(\ln \rho)/Dt$ from (9) and plug it into (2), get:

$$\boxed{\frac{1}{\gamma}\frac{D \ln p}{Dt} + \frac{\partial u}{\partial x} = 0}. \tag{10}$$

The last step doesn't completely remove ρ itself from problem because it's still in (1):

$$\boxed{\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}}. \tag{1}$$

But if we treat ρ as constant in (1) then (10) and (1) count as 2 equations in 2 unknowns (*u* and *p*). This will be our approach.