LECTURE 39

Sound waves (continued)

Our governing equations are now just the x-comp equation of motion,

$$\boxed{\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}}. \tag{1}$$

and mass conservation (in which thermo energy equation for adiabatic motion, ideal gas law, and definition of potential temp knocked $\ln\rho$ out in favor of $\ln p$).

$$\boxed{\frac{1}{\gamma}\frac{D\ln p}{Dt} + \frac{\partial u}{\partial x} = 0}. \tag{10}$$

Linearizing the governing equations

Consider a reference atmosphere (atmosphere before wave arrives) in which u = ***U* (constant), *v* = *w* = 0, density is constant, ρ_c, and pressure is hydrostatic, *P*(*z*)**. In presence of the wave we write:

$$u(x,t) = U + u'(x,t), \qquad \rho(x,t) = \rho_c + \rho'(x,t), \qquad p(x,t) = P(z) + p'(x,t),$$

where a prime denotes a perturbation variable. Perturbations are small in sense that: $|u'| << |U|$, $|\rho'| << \rho_c$, $|p'| << P$. Expand out Du/Dt in x-eqn of motion (1), get:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{Plug in } u = U + u',\ p = P + p', \text{etc}$$

$$\underbrace{\boxed{\frac{\partial U}{\partial t}}}_{\substack{\downarrow \\ 0 \\ \text{since U is const}}} + \frac{\partial u'}{\partial t} + \underbrace{\boxed{U\frac{\partial U}{\partial x}}}_{\substack{\downarrow \\ 0 \\ \text{since U is const}}} + U\frac{\partial u'}{\partial x} + \underbrace{\boxed{u'\frac{\partial U}{\partial x}}}_{\substack{\downarrow \\ 0 \\ \text{since U is const}}} + u'\frac{\partial u'}{\partial x} = -\underbrace{\boxed{\frac{1}{\rho_c + \rho'}}}_{\substack{\downarrow \\ \text{use Binomial approx}}}\left(\underbrace{\boxed{\frac{\partial P}{\partial x}}}_{\substack{\downarrow \\ 0 \\ \text{since } P \text{ is indep of } x}} + \frac{\partial p'}{\partial x}\right)$$

from Binomial approx: $\quad \dfrac{1}{\rho_c+\rho'} = \dfrac{1}{\rho_c\left(1+\dfrac{\rho'}{\rho_c}\right)} = \dfrac{1}{\rho_c}\left(1+\dfrac{\rho'}{\rho_c}\right)^{-1} \approx \dfrac{1}{\rho_c}\left(1-\dfrac{\rho'}{\rho_c}\right)$

$$\therefore \quad \frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x} = -\frac{1}{\rho_c}\left(1-\frac{\rho'}{\rho_c}\right)\frac{\partial p'}{\partial x}$$

Now **linearize this equation: throw out the non-linear terms** (products of perturbations). These products are really tiny (since the perturbations are tiny) so we can safely neglect them. The resulting equation is **linear**.

$$\therefore \quad \frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} = -\frac{1}{\rho_c}\frac{\partial p'}{\partial x}$$

$$\therefore \quad \boxed{\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)u' = -\frac{1}{\rho_c}\frac{\partial p'}{\partial x}}. \tag{11}$$

Now work with (10). Set it up for easy linearization by reverting from ln*p* back to *p*:

$$\frac{1}{\gamma}\frac{1}{p}\frac{Dp}{Dt} + \frac{\partial u}{\partial x} = 0 \qquad \text{mult by } \gamma\,\text{p}$$

$$\frac{Dp}{Dt} + \gamma p\frac{\partial u}{\partial x} = 0 \qquad \text{Expand out total deriv}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} = 0 \qquad \text{Plug in } u = U + u',\ p = P + p'\text{, etc}$$

$$\underset{\substack{\downarrow\\ 0\\ \text{P is indep of t}}}{\boxed{\frac{\partial P}{\partial t}}} + \frac{\partial p'}{\partial t} + \underset{\substack{\downarrow\\ 0\\ \text{P is indep of x}}}{\boxed{U\frac{\partial P}{\partial x}}} + U\frac{\partial p'}{\partial x} + \underset{\substack{\downarrow\\ 0\\ \text{P is indep of x}}}{\boxed{u'\frac{\partial P}{\partial x}}} + u'\frac{\partial p'}{\partial x} + \gamma\left(P+p'\right)\left(\underset{\substack{\downarrow\\ 0\\ U\text{ is const}}}{\boxed{\frac{\partial U}{\partial x}}} + \frac{\partial u'}{\partial x}\right) = 0$$

$$\frac{\partial p'}{\partial t} + U\frac{\partial p'}{\partial x} + u'\frac{\partial p'}{\partial x} + \gamma\left(P+p'\right)\frac{\partial u'}{\partial x} = 0$$

Now **linearize it: throw out the non-linear terms**

$$\frac{\partial p'}{\partial t}+U\frac{\partial p'}{\partial x}+\gamma P\frac{\partial u'}{\partial x}=0\,.$$

$$\boxed{\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)p'+\gamma P\frac{\partial u'}{\partial x}=0}\,. \tag{12}$$

Wave solutions of the linearized governing equations

Equations (11), (12) are 2 equations in 2 unknowns (p', u'). Combine them to get 1 equation for just p' or u'. Okay, lets get rid of u'. In (12) u' shows up only as $\partial u'/\partial x$. We want to "create" a similar term in (11). So take $\partial/\partial x$ (11):

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\boxed{\frac{\partial u'}{\partial x}}=-\frac{1}{\rho_c}\frac{\partial^2 p'}{\partial x^2}$$ Now plug in $\frac{\partial u'}{\partial x}$ from (12) (watch minus sign)

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\boxed{\frac{\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)p'}{-\gamma P}}=-\,\frac{1}{\rho_c}\frac{\partial^2 p'}{\partial x^2}$$ Mult by $-\gamma P$ and rearrange

$$\boxed{\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2 p'-\frac{\gamma P}{\rho_c}\frac{\partial^2 p'}{\partial x^2}=0},$$ **Wave equation for sound waves** (13)

where we've used special "symbolic" notation for the first term:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2 p'=\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)p'=\left(\frac{\partial^2}{\partial t^2}+2U\frac{\partial^2}{\partial x\partial t}+U^2\frac{\partial^2}{\partial x^2}\right)p'\,.$$

Consider a trial solution of (13) as a sine or cosine (your choice). Okay, work with

$$p'=A\cos(kx-\omega t)\,. \tag{14}$$

Plug (14) into (13). Pre-calculate some of the derivatives:

$$\frac{\partial p'}{\partial x} = -Ak\sin(kx-\omega t)$$

$$\frac{\partial^2 p'}{\partial x^2} = -Ak^2\cos(kx-\omega t)$$

$$\frac{\partial p'}{\partial t} = -A(-\omega)\sin(kx-\omega t) = A\omega\sin(kx-\omega t)$$

$$\frac{\partial^2 p'}{\partial t^2} = -A\omega^2\cos(kx-\omega t)$$

$$\frac{\partial^2 p'}{\partial x \partial t} = Ak\omega\cos(kx-\omega t)$$

So (13) becomes (careful with signs!):

$$-A\omega^2\cos(kx-\omega t) + 2UAk\omega\cos(kx-\omega t) - U^2Ak^2\cos(kx-\omega t) \\ + \frac{\gamma P}{\rho_c}Ak^2\cos(kx-\omega t) = 0$$

Divide through by common factor of $A\cos(kx-\omega t)$:

$$-\omega^2 + 2Uk\omega - U^2k^2 + \frac{\gamma P}{\rho_c}k^2 = 0.$$

Combine first 3 terms (they came from $\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 p'$ so we're just collapsing them back to their original form):

$$-\left(\omega - Uk\right)^2 + \frac{\gamma P}{\rho_c}k^2 = 0.$$

To find ω, rearrange above equation and take square root.

$$\left(\omega - Uk\right)^2 = \frac{\gamma P}{\rho_c}k^2$$

$$\omega - Uk = \pm k\sqrt{\frac{\gamma P}{\rho_c}}$$

$$\omega = kU \pm k\sqrt{\frac{\gamma P}{\rho_c}}$$

Relating P/ρ_c to reference temp T through ideal gas law, $P = \rho_c RT$, ω becomes

$$\omega = kU \pm k\sqrt{\gamma RT} \qquad \textbf{Dispersion relation for sound waves.} \tag{15}$$

So, $p' = A\cos(kx - \omega t)$ is a solution of the wave equation provided ω satisfies (15).

A dispersion relation tells us how ω varies with k and the physical parameters for a particular wave type (sound waves in this case). Dispersion relations are derived from dynamical and thermodynamical equations. In contrast, $\omega = ck$ or equivalently $c = \omega/k$, arose from kinematic considerations, by analyzing the generic form of the wave phase. They are not dispersion relations.

Dispersion relations describe (implicitly) how waves of different wavenumber propagate with respect to each other. To make that explicit, use dispersion relation in $c = \omega/k$ to get c. For sound waves:

$$\boxed{c = U \pm \sqrt{\gamma RT}}\,. \tag{16}$$

No k in (16), so waves propagate with same phase speed regardless of wavelength. **The dispersion relation tells us that sound waves are non-dispersive.** [Good thing, too. If this wasn't the case, humans couldn't use sound to communicate.]

We can rewrite (16) as

$$c = U \pm c_s$$

where $c_s \equiv \sqrt{\gamma RT}$, (17) is **adiabatic speed of sound**. Under ordinary conditions at sea level, $c_s \approx 350\,\mathrm{m\,s^{-1}}$.

The $\pm$ ambiguity arises because 1D waves can propagate away from source toward right ($+x$) or left ($-x$). U can also be $+$ or $-$. Consider the frequency $\omega = k\left(U \pm \sqrt{\gamma RT}\right)$ heard by an observer under different conditions:

If **wind is westerly (*U* > 0) and observer is to right of source**, observer hears frequency of the rightward moving wave $+k\sqrt{\gamma RT}$ but **increased** by a **positive Doppler shift *kU***. An equivalent situation occurs if there is no wind but the sound source (e.g., an ambulance) moves to the right at speed U, i.e., toward observer.

If **wind is easterly (*U* < 0) and observer is still to right of source**, observer hears frequency of the rightward moving wave $+k\sqrt{\gamma RT}$ but **decreased** by a **negative Doppler shift *kU*** (negative since $U < 0$). An equivalent situation occurs if there is no wind but the sound source moves to the left, i.e., away from observer.