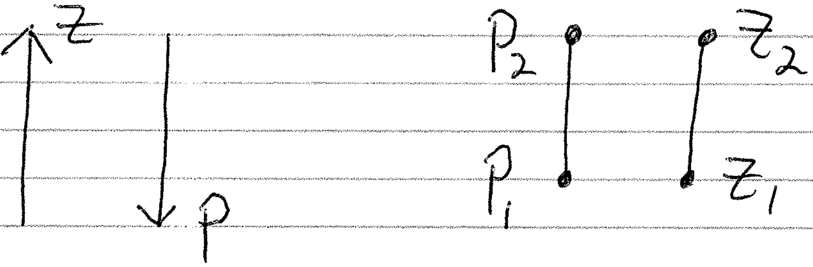
Lecture 4

Review of the hypsometric equation.

Consider two pressure levels at a fixed x,y location.



Let the lower altitude pressure be p_1
and " upper " " " p_2.

$z_2 > z_1$
$p_2 < p_1$

Lets relate the difference $z_2 - z_1$ to the pressures p_1 and p_2.

$$dp = \frac{\partial p}{\partial z} dz$$ Use hydrostatic equation for $\frac{\partial p}{\partial z}$

$$= -\rho g \, dz$$ Now use ideal gas law $p = \rho R T \rightarrow \rho = \frac{p}{RT}$

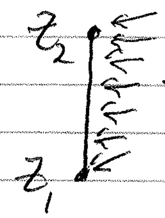
$$= -\frac{pg}{RT} dz$$ Rearrange to get expression for dz

$$dz = -\frac{RT}{g} \frac{dp}{p}$$ Now integrate btw the 2 levels.

$$\int_{z_1}^{z_2} dz' = -\frac{R}{g}\int_{p_1}^{p_2} T \frac{dp'}{p'}$$

$\frac{dp'}{p'} \rightarrow d\ln p'$

a prime denotes a dummy variable



$$\int_{z_1}^{z_2} dz' = -\frac{R}{g}\int_{\ln p_1}^{\ln p_2} T\, d(\ln p')$$

$$\left[z'\right]\Big|_{z_1}^{z_2}$$

Use fact that Ave = $\frac{\text{integral}}{\text{interval}}$

So integral = Ave times interval

$$\therefore \int_{\ln p_1}^{\ln p_2} T\, d\ln p' = \overline{T}(\ln p_2 - \ln p_1)$$

$$\therefore z_2 - z_1 = -\frac{R}{g}\overline{T}(\ln p_2 - \ln p_1)$$

use: $-(\ln p_2 - \ln p_1) = -\ln \frac{p_2}{p_1}$

$= \ln \frac{p_1}{p_2}$

Since $p_1 > p_2$: $\frac{p_1}{p_2} > 1$

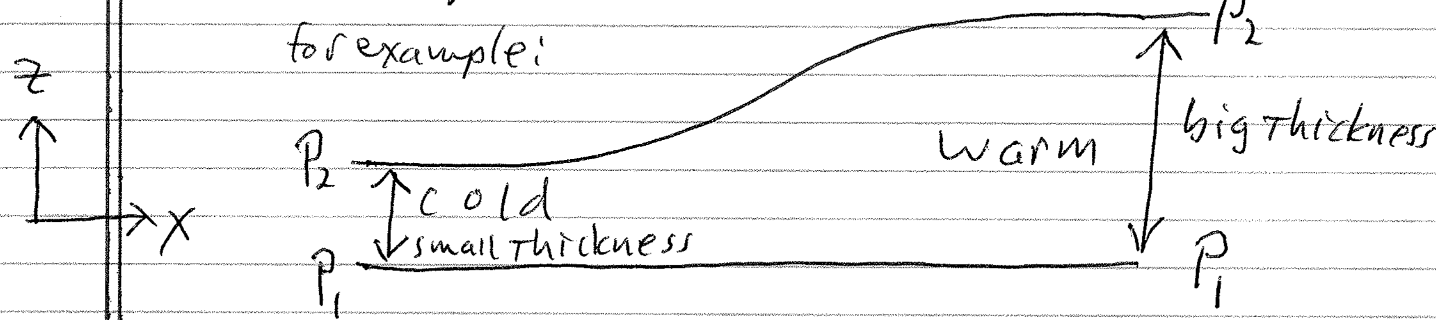
so $\ln \frac{p_1}{p_2}$ is positive.

$$\therefore \boxed{z_2 - z_1 = \frac{R}{g}\ln\left(\frac{p_1}{p_2}\right)\overline{T}}$$

Hypsometric equation.

$z_2 - z_1$ is the thickness of the atmosphere btw pressure levels p_1 and p_2. The thickness is proportional to the temperature in the layer.

for example:



Can rewrite hypsometric eq^n in terms of geopotential Φ (recall $\Phi = gz$).

$$\Phi_2 - \Phi_1 = R \ln\left(\frac{p_1}{p_2}\right)\overline{T}$$

$$\boxed{\text{Review of isobaric coordinates}}$$

big → assumption In cases where hydrostatic equation is a good approximation (to full vertical equation of motion), it's often convenient to work with pressure as the "vertical" coordinate rather than z.

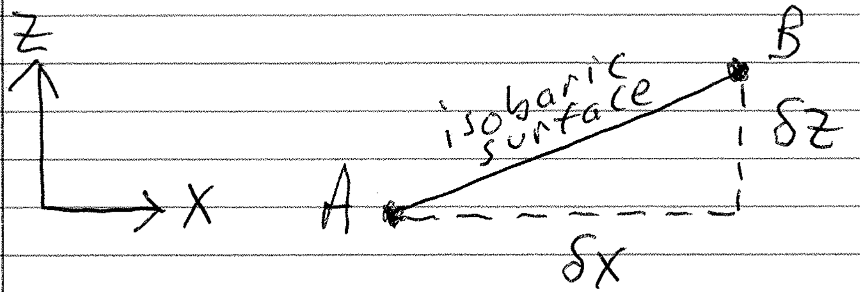
Isobaric coordinates: x, y, p
Cartesian " : x, y, z

How do we express the horizontal p.g.f. in isobaric coordinates?

In Cartesian coords: $-\frac{1}{\rho}\frac{\partial p}{\partial x}$

which means $-\frac{1}{\rho}\left.\frac{\partial p}{\partial x}\right|_{y,z}$ ← y, z fixed while x varies

Want to rewrite it in isobaric coordinates. Consider a tiny xz cross section through an isobaric surface:



Solid line is a line of const pressure.
So $P_A = P_B$
Slope of line is $\frac{\delta z}{\delta x}$

$$P_B = P_A + \left.\frac{\partial P}{\partial x}\right|_{\substack{\text{at } A \\ y,z \text{ const}}} \delta x + \left.\frac{\partial P}{\partial z}\right|_{\substack{\text{at } A \\ x,y \text{ const}}} \delta z + \underbrace{\text{h.o.t.}}_{\substack{\text{vanish} \\ \text{as } \delta x \\ \text{and } \delta z \to 0}}$$

Use fact that $P_B = P_A$

$$\therefore\ 0 = \frac{\partial P}{\partial x}\delta x + \frac{\partial p}{\partial z}\delta z$$ $\div\ \delta x$ and rearrange

$$\frac{\partial p}{\partial x} = -\frac{\partial p}{\partial z}\frac{\delta z}{\delta x}$$ use hydrostatic equation $\frac{\partial p}{\partial z} = -\rho g$

$$\therefore\ \frac{\partial p}{\partial x} = \rho g \frac{\delta z}{\delta x}$$ Take $\delta x \to 0$ and use:

$$\lim_{\delta x \to 0} \frac{\delta z}{\delta x} = \frac{\partial z}{\partial x}$$

$$\therefore\ \frac{\partial p}{\partial x} = \rho g \frac{\partial z}{\partial x}$$

This means $\left.\frac{\partial p}{\partial x}\right|_z = \rho g \left.\frac{\partial z}{\partial x}\right|_p$

z fixed   p fixed

To get p.g.f. mult by $-\frac{1}{\rho}$:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}\Big|_z = -g\frac{\partial z}{\partial x}\Big|_p$$

Rewrite using $\Phi = gz$:

(1) $$-\frac{1}{\rho}\frac{\partial p}{\partial x}\Big|_z = -\frac{\partial \Phi}{\partial x}\Big|_p$$

Similarly, by considering yz cross section we get

(2) $$-\frac{1}{\rho}\frac{\partial p}{\partial y}\Big|_z = -\frac{\partial \Phi}{\partial y}\Big|_p$$

Take $\hat{i}$ times (1) + $\hat{j}$ times (2), get:

(3) $$-\frac{1}{\rho}\nabla_z p = -\nabla_p \Phi$$

Often we'll write (1), (2) or (3) with the subscripts z and p left off, but their existence is <u>implied</u>.

Note that the <u>left hand sides</u> of these eq^{ns} (Cartesian forms) <u>contain two dependent variables</u>: p and ρ. But the <u>right hand sides</u> (isobaric coordinate forms) contain only <u>one</u> <u>dependent</u> variable: just p.

So these isobaric forms are <u>simpler</u>!

Geostrophic wind also simplifies in isobaric coordinates:

In Cartesian coords: $\vec{U}_g = \frac{1}{\rho f} \hat{k} \times \nabla p$

In isobaric coords: $\vec{U}_g = \frac{1}{f} \hat{k} \times \nabla \Phi$

Take $\nabla_p \cdot \vec{U}_g$ (horiz div on isobaric sfc):

$$\nabla_p \cdot \vec{U}_g = \nabla_p \cdot \frac{1}{f} \hat{k} \times \nabla \Phi = \frac{1}{f} \nabla_p \cdot (\hat{k} \times \nabla \Phi)$$

neglect latitudinal variation in f

$$\hat{k} \times \nabla \Phi = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 1 \\ \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} & 0 \end{vmatrix} = -\hat{\imath} \frac{\partial \Phi}{\partial y} + \hat{\jmath} \frac{\partial \Phi}{\partial x}$$

evaluated on isobaric sfc

$$\therefore \nabla_p \cdot \vec{U}_g = \frac{1}{f}\left(\hat{\imath} \frac{\partial}{\partial x} + \hat{\jmath} \frac{\partial}{\partial y}\right) \cdot \left(-\hat{\imath} \frac{\partial \Phi}{\partial y} + \hat{\jmath} \frac{\partial \Phi}{\partial x}\right)$$

$$= \frac{1}{f}\left(-\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial y} + \boxed{\frac{\partial}{\partial y}\frac{\partial \Phi}{\partial x}}\right)$$ interchange order of differentiation

$$= \frac{1}{f}\left(-\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial y}\right)$$

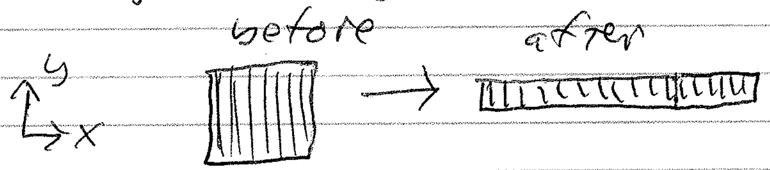
$$= 0$$

$$\therefore \quad \boxed{\nabla_p \cdot \vec{U}_g = 0}$$

Geostrophic wind is quasi-horizontally non-divergent, i.e. x, y divergence of geos wind evaluated on an isobaric sfc is 0.

So airparcels moving with the geostrophic wind cannot change their horizontal area ($\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{A}\frac{DA}{Dt}$). Horiz area of parcel is conserved.

So, a parcel moving with geostrophic wind can do this:



but NOT this!