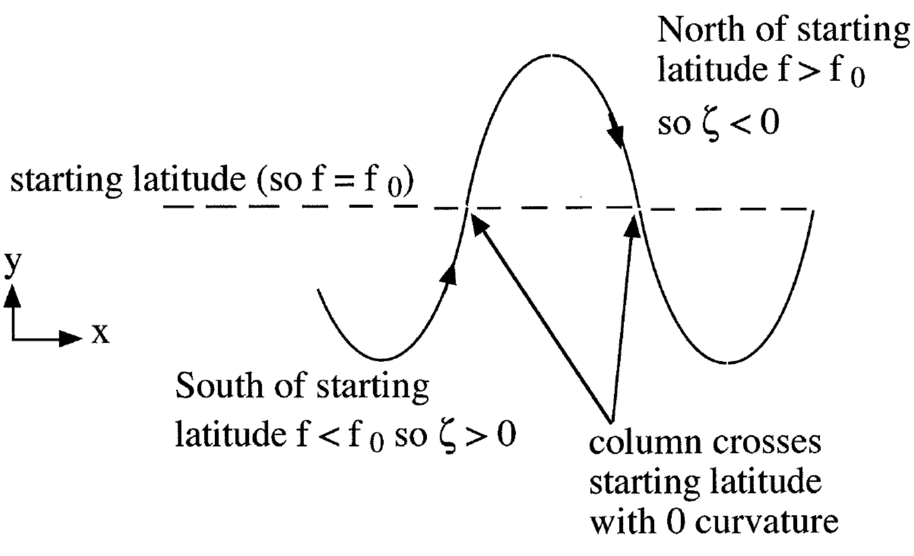
LECTURE 40

Rossby waves (Planetary waves)

Based on our analysis of an air column (with $\zeta = 0$ initially) passing over a mountain ridge (Lecture 23), we suspect **latitudinal variations in *f* can support wave motions**. When the column reached the plains, it was south of its starting latitude, and moving north with $\zeta > 0$. Assuming column thickness H didn't change over the plains, the barotropic potential vorticity theorem $\frac{D_h}{Dt}\left(\frac{\zeta+f}{H}\right)=0$ reduced to $\frac{D_h}{Dt}\left(\zeta+f\right)=0$, so the column's absolute vertical vorticity was conserved. Increases in f were associated with decreases in ζ (and vice versa). So this happened:



Now see if we can get propagating wave solutions due to latitudinal changes in f. Consider the **vertical vorticity equation for mid-latitude synoptic-scale flows**,

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}=-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\zeta+f\right)-\beta v. \tag{1}$$

with $\beta \equiv df/dy = 2\Omega\cos\phi/a > 0$. We'll treat β as const. Recall that $-\beta v$ accounts for advection of earth vorticity (f).

Consider the simple case where the flow is **2D** [$u = u(x, y, t)$, $v = v(x, y, t)$, $w(x, y, t) = 0$] and **incompressible**. In 2D, $\nabla \cdot \vec{u} = 0$ becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2}$$

This knocks out the stretching term in (1), and we get the vorticity equation as

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = -\beta v. \tag{3}$$

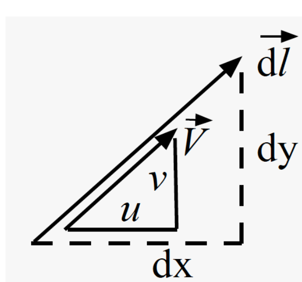
Also, for flows satisfying (2) we can work with a single scalar ψ (streamfunction) rather than u and v separately. ψ is defined indirectly by:

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}. \tag{4}$$

Note that for u and v that satisfy (4), then (2) is automatically satisfied:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial^2 \psi}{\partial x \partial y} + \underbrace{\boxed{\frac{\partial^2 \psi}{\partial y \partial x}}}_{\text{interchange order of differentiation}} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0.$$

ψ also has a nice graphical property: **lines of constant ψ are streamlines**. To show this, we'll show that the slopes of the two lines are the same. Let $\vec{dl} = \hat{i}\,dx + \hat{j}\,dy$ be a chunk of a line of constant ψ. So $d\psi = 0$ on that line. Using chain rule, this expands out to $\partial\psi/\partial x\ dx + \partial\psi/\partial y\ dy = 0$, or, using (4): $v\ dx - u\ dy = 0$. So the slope of this line is $dy/dx = v/u$. But slope of streamline passing through the same point is also v/u. So lines of constant ψ and streamlines coincide.



Can write vorticity ζ in terms of ψ as:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla_h^2 \psi .$$

So the vorticity equation (3) becomes

$$\frac{\partial}{\partial t}\left(\nabla_h^2 \psi\right) - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla_h^2 \psi\right) + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\nabla_h^2 \psi\right) = -\beta \frac{\partial \psi}{\partial x}. \quad (5)$$

Consider a reference atmosphere in which the wind is a uniform westerly current, U (> 0). Since U is constant (and v is 0), the reference atmosphere ζ is 0. In view of (4), ψ in the reference atmosphere is $-U\,y$.

Decompose ψ into reference atmosphere and perturbation parts:

$$\psi = -U\,y + \psi' .$$

Applying this in (5) yields

$$\frac{\partial}{\partial t}\left(\nabla_h^2 \psi'\right) + \left(U - \frac{\partial \psi'}{\partial y}\right)\frac{\partial}{\partial x}\left(\nabla_h^2 \psi'\right) + \frac{\partial \psi'}{\partial x}\frac{\partial}{\partial y}\left(\nabla_h^2 \psi'\right) = -\beta \frac{\partial \psi'}{\partial x}.$$

Now linearize it! Get:

$$\frac{\partial}{\partial t}\left(\nabla_h^2 \psi'\right) + U \frac{\partial}{\partial x}\left(\nabla_h^2 \psi'\right) = -\beta \frac{\partial \psi'}{\partial x}. \quad (6)$$

Seek wavy solutions of (6) of the form

$$\psi' = A\cos\left(kx - \omega t\right). \quad (7)$$

Pre-calculate some of the derivatives (be really careful with the signs!!!)

$$\frac{\partial \psi'}{\partial x} = -Ak\sin(kx - \omega t)$$

$$\frac{\partial^2 \psi'}{\partial x^2} = -Ak^2\cos(kx - \omega t)$$

and since $\frac{\partial \psi'}{\partial y} = 0$, $\frac{\partial^2 \psi'}{\partial y^2} = 0$, we see that

$$\nabla_h^2 \psi' = -Ak^2 \cos(kx - \omega t). \text{ So}$$

$$\frac{\partial}{\partial x} \nabla_h^2 \psi' = (-Ak^2) \frac{\partial}{\partial x} \cos(kx - \omega t) = (-Ak^2)(-k)\sin(kx - \omega t)$$
$$= Ak^3 \sin(kx - \omega t)$$

$$\frac{\partial}{\partial t} \nabla_h^2 \psi' = (-Ak^2) \frac{\partial}{\partial t} \cos(kx - \omega t) = (-Ak^2)(-\omega)\left[-\sin(kx - \omega t)\right]$$
$$= -Ak^2 \omega \sin(kx - \omega t)$$

The vorticity equation (6) now becomes (careful with signs!!!),

$$-Ak^2 \omega \sin(kx - \omega t) + UAk^3 \sin(kx - \omega t) = \beta Ak \sin(kx - \omega t)$$

Divide by common factor $Ak \sin(kx - \omega t)$, get

$$-k\omega + Uk^2 = \beta$$

Solve for ω:

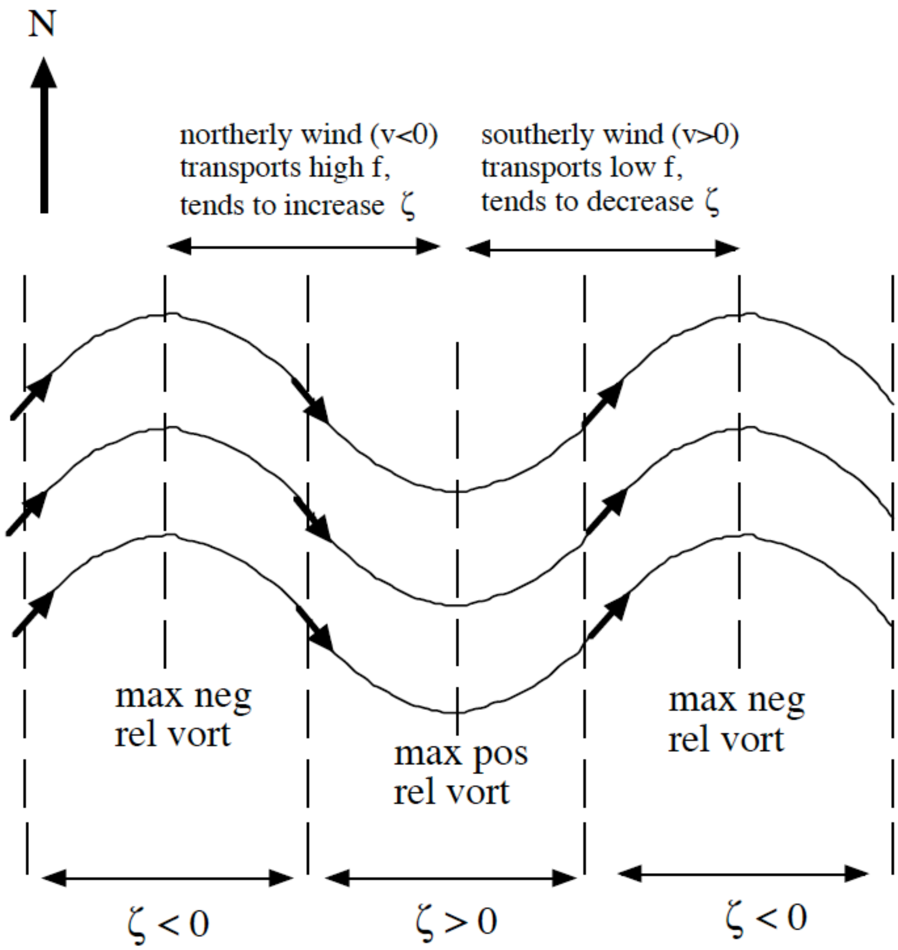
$$\boxed{\omega = Uk - \frac{\beta}{k}}. \qquad \textbf{Dispersion relation for Rossby waves} \qquad (8)$$

Apply this in $c = \frac{\omega}{k}$ to get the Rossby wave phase speed:

$$\boxed{c = U - \frac{\beta}{k^2}} \quad \text{or, in terms of wavelength:} \quad \boxed{c = U - \beta\left(\frac{\lambda}{2\pi}\right)^2}$$

U (which advects relative vorticity toward east) tries to make wave propagate toward east, while β term tries to make wave propagate toward west (retrogress). For short waves (λ small; k big), U wins so $c > 0$ and wave propagates eastward. For long waves (λ small; k big), β wins and wave retrogresses.

How does β term in vorticity eqn try to make wave retrogress? Consider contour plot of $\psi = -U y + \psi'$, keeping in mind that lines of constant ψ are streamlines:



In northerlies:

$\partial\zeta/\partial x > 0$ (ζ **increases toward east**), while earth vort advection yields $\partial\zeta/\partial t > 0$ (ζ **increases with *t***). As a result, ζ pattern and associated ψ pattern appear to shift westward. Actually, those patterns really do shift westward: **patterns (geometrical thing**) shifts westward while **parcels (physical thing**) have an eastward component.

In southerlies:

$\partial\zeta/\partial x < 0$ (ζ **decreases toward east**), while earth vort advection yields $\partial\zeta/\partial t < 0$ (ζ **decreases with *t***). As a result, the ζ pattern and associated ψ pattern again shift westward.