Review of 1st order linear ordinary differential equations

METR 3123, Atmospheric Dynamics II
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Suggested reading/reference:

Braun, M., 1992: *Differential Equations and Their Applications*. 4th edition. Springer-Verlag.

- There are two books: a full book and a "short edition". Don't get the short edition. The full book is clear, and packed with interesting examples: famous art forgeries, growth of populations, vibrations leading to the Tacoma bridge disaster, atomic waste disposal, spread of plague in Bombay, predator-prey problems, mathematical theories of war, and even the spread of gonorrhea! Wow!!!

Today we'll cover terminology and classification of general d.e.s, then we'll focus on 1st order linear odes.

A differential equation (d.e.) is an eq^n that involves derivatives of a dependent variable (unknown) with respect to one or more independent variables.

If only one indep variable then the d.e. is an ordinary differential equation (o.d.e.)

If more than one indep variable then the d.e. is a partial differential equation (p.d.e.)

e.g.: $\dfrac{d^2T}{dx^2} = 0$

T is dependent variable.
x is independent variable.
∴ eq^n is an o.d.e. [note d/dx notation for derivs]

e.g.: $\dfrac{\partial^2T}{\partial x^2} + \dfrac{\partial^2T}{\partial y^2} + \dfrac{\partial^2T}{\partial z^2} = 0$

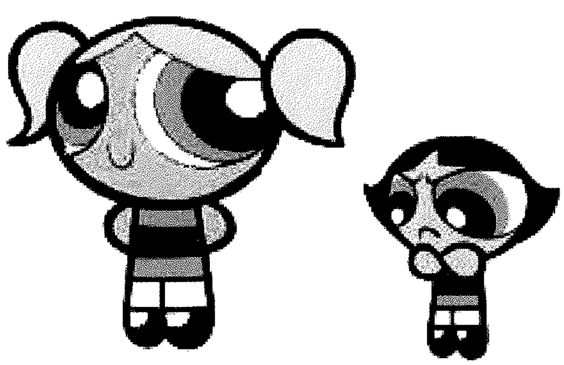
T is the dependent variable.
x, y and z are the indep variables (3 of 'em)
∴ eq^n is a p.d.e. [note $\partial/\partial x$ notation for x derivs]

Today we'll look at odes (mostly 1^{st} order linear odes)

The order of a d.e. is the order of the highest derivative of the dependent variable. Previous 2 examples were 2^{nd} order d.e.s.

Linear versus nonlinear o.d.e.s: If dependent variable or its derivatives appear to the first power or zeroth power (i.e. doesn't show up) in every term then the o.d.e. is linear. Otherwise the o.d.e. is nonlinear. Our previous 2 examples were linear.

Let's look at more examples.



You brown-noser.

e.g.: $$t^2 \boxed{\frac{d^2y}{dt^2}} + \sin t \boxed{\frac{dy}{dt}} + (\cos^2 t)\boxed{y} = 3t$$

2nd deriv of y to <u>first</u> <u>power</u>	1st deriv of y to <u>first power</u>	y itself to <u>first</u> <u>power</u>	y to <u>zeroth</u> <u>power</u> (i.e., no y).

dependent variable is y
1 independent variable: t
so it's an ode
2nd order ode (since highest deriv of dep variable is 2).
y and it's derivs appear at most to 1st power so: <u>linear</u>
But coeffs are variable.

e.g.: $\frac{d^2y}{dt^2} + y = e^t$ 2nd order linear ode w/ var coeff

e.g.: $x^5 \frac{d^2z}{dx^2} + (\tan x)\, z = \sin x\, e^x$ 2nd order linear ode w/ var coeff

e.g.: $\frac{d^2y}{dx^2} + 1 = \tan y$ 2nd order nonlinear ode

e.g.: $\frac{d^2y}{dx^2} + y^2 = 0$ 2nd order nonlinear ode

e.g.: $\frac{dy}{dx} + y^2 = 0$ 1st order nonlinear ode

<u>In a linear eq$^{\underline{n}}$</u>, if there's a term <u>without</u> the dependent variable in it then the eq^n is <u>inhomogeneous</u> (nonhomogeneous). If every term has the dependent variable in it then the equation is <u>homogeneous</u>.

Some examples of linear homogeneous odes:

$$\frac{dy}{dx} + x^5 y = 0$$

$$\frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} = x^5 y$$

$$\frac{d^3y}{dx^3} = y$$

Some examples of linear inhomogeneous odes:

$$\frac{dy}{dx} + x^5 y + 1 = 0$$

$$\frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} + x^3 = x^5 y$$

$$\frac{d^3y}{dx^3} = x$$

OK, now let's focus on 1st order linear o.d.e.s

(I) Simplest case: $\boxed{\frac{dy}{dx} = f(x)}$

"deriv of unknown = known"

f(x) can be anything:

$$\begin{aligned} \text{e.g. } f(x) &= 1 \\ &= x \\ &= x \sin x \, J_0(x) \quad \text{or whatever} \end{aligned}$$

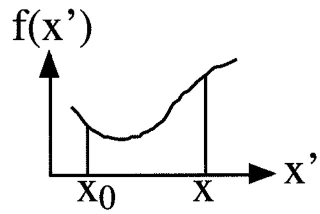
Multiply ode by dx

$dy = f(x)\,dx$ [we've separated variables. All y stuff on one side, all x stuff on the other side.]

Now integrate it. Call the lower limit of integration x_0. Let y_0 be the value of y when $x = x_0$, that is: $y_0 = y(x_0)$

$$\int_{y_0}^{y} dy' = \int_{x_0}^{x} f(x')\,dx'$$

x' and y' are dummy variables



$$\therefore \quad [y']\Big|_{y_0}^{y} = \int_{x_0}^{x} f(x')\,dx'$$

$$\therefore \quad y - y_0 = \int_{x_0}^{x} f(x')\,dx'$$

$$\therefore \quad \boxed{y = y_0 + \int_{x_0}^{x} f(x')\,dx'}$$

You can also write the solution as the indefinite integral:

$$y = \int^{x} f(x')\,dx' + C$$

Next simplest case:

(II) $\boxed{\dfrac{dy}{dx} + f(x)\,y = 0}$

"deriv of unknown = known times unknown"

- Most general 1st order linear homogeneous ode.

- characteristic of radioactive decay problems, growth of bacteria, radiative cooling of air parcels,

÷ ode by y

$$\boxed{\frac{1}{y}\frac{dy}{dx}} + f(x) = 0$$

$$\downarrow$$

$$\frac{d\ln y}{dx}$$

$$\therefore \quad \frac{d\ln y}{dx} = -f(x)$$

Now it's essentially the same as case I! Multiply by dx and integrate.

$$\boxed{\int_{y_0}^{y} d\ln y'} = -\int_{x_0}^{x} f(x')\,dx'$$

$$\downarrow$$

$$\left[\ln y'\right]\Big|_{y_0}^{y} = \ln y - \ln y_0 = \ln\frac{y}{y_0}$$

$$\therefore \quad \ln\frac{y}{y_0} = -\int_{x_0}^{x} f(x')\,dx'$$ exponentiate it:

$$\therefore \quad e^{\ln(y/y_0)} = e^{-\int_{x_0}^{x} f(x')\,dx'}$$

$$\therefore \quad \frac{y}{y_0} = e^{-\int_{x_0}^{x} f(x')\,dx'}$$ multiply by y_0:

$$\therefore \quad \boxed{y = y_0 e^{-\int_{x_0}^{x} f(x')\,dx'}}$$

Another way to solve it (with indefinite integration):

$$\frac{d\ln y}{dx} = -f(x)$$

$d\ln y = -f(x)\,dx$ use indefinite integration, get:

$\ln y = -\int^{x} f(x')\,dx' + \text{const}$ exponentiate it:

$$y = e^{[-\int^{x} f(x')\,dx' + \text{const}]}$$

$y = e^{-\int^{x} f(x')\,dx'}\, e^{\text{const}}$ let $C = e^{\text{const}}$

$$y = C e^{-\int^{x} f(x')\,dx'}$$

e.g. solve $\dfrac{dy}{dx} + x^2 y = 0$ <u>subject to y(1) = 3.</u>

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = -x^2$$

$$\therefore \quad \frac{d\ln y}{dx} = -x^2$$

indefinite integration yields:

$\ln y = -x^3/3 + \text{const}$ exponentiating it yields:

$$\therefore \quad y = e^{(-x^3/3 + \text{const})} = e^{-x^3/3}\, \boxed{e^{\text{const}}} \longrightarrow C$$

$$\therefore \quad y = C e^{-x^3/3}$$

To calculate C use fact that y(1) = 3:

$$3 = C\,e^{-1/3}$$

So $C = 3\,e^{1/3}$

So $y = 3\,e^{1/3}\,e^{-x^3/3}$ or

$$y = 3\exp\left[-\frac{1}{3}\left(x^3 - 1\right)\right]$$

Now solve the exact same problem, but this time use <u>definite integration</u>. As before, get:

$$\frac{d\ln y}{dx} = -x^2$$

definite integration yields:

$$\int_3^y d\ln y' = -\int_1^x x'^2\,dx'$$

$$\therefore \quad \left[\ln y'\right]\Big|_3^y = -\left[\frac{x'^3}{3}\right]\Bigg|_1^x$$

$$\therefore \quad \ln y - \ln 3 = -\frac{1}{3}\left(x^3 - 1\right)$$

$$\therefore \quad \ln(y/3) = -\frac{1}{3}\left(x^3 - 1\right)$$ exponentiate it

$$\therefore \quad y/3 = \exp\left[-\frac{1}{3}\left(x^3 - 1\right)\right]$$ multiply by 3

$$\therefore \quad y = 3\exp\left[-\frac{1}{3}\left(x^3 - 1\right)\right] \quad \text{as before.}$$

Now consider the more general problem (still linear though):

$$\text{(III)} \quad \boxed{\frac{dy}{dx} + f(x)\,y = g(x)}$$

"deriv of unknown = known + known times unknown"

1st order linear inhomogeneous ode
(most general 1st order linear ode)

÷ by y helped before but now it won't. We need another trick.
It would be nice if (III) could be rewritten as:

$$\frac{d}{dx}\left(\text{unknown times known}\right) = \text{known}$$

because then it would be like (I) and we could just integrate it. It turns out we can do that if we first multiply (III) by a "useful" function $\mu(x)$. μ will be <u>defined</u> to be whatever function that lets us write (III) in this nice form. Accordingly, μ is called an <u>integrating factor</u>.

Multiply (III) by μ (we don't know what it is just yet):

$$(*) \quad \boxed{\mu \frac{dy}{dx} + \mu f y} = \mu g$$

$$\downarrow$$

Want to rewrite (*) as

$$(**) \quad \boxed{\frac{d(\mu y)}{dx}} = \mu g$$

That is, want to find μ such that:

$$\frac{d(\mu y)}{dx} = \mu\frac{dy}{dx} + \mu f y$$

Use the product rule to expand the l.h.s. of this eq^n, get:

$$\mu\frac{dy}{dx} + y\frac{d\mu}{dx} = \mu\frac{dy}{dx} + \mu f y \qquad \text{get some cancellation}$$

$$\therefore \quad y\frac{d\mu}{dx} = \mu\, f\, y$$

y is common to both terms so ÷ through by it, get:

$$\frac{d\mu}{dx} = \mu\, f$$

It's a 1st order linear homogeneous ode (for μ): case (II)!
÷ by μ and integrate -- definite or indefinite integration, it doesn't matter. [Note: An infinite number of μ's are possible, any one of which is fine. If you use indefinite integration then μ depends on what you choose for the constant of integration. If you use definite integration then μ depends on the value μ_0 you choose to correspond to $x = x_0$.] Let's get μ both ways:

Definite integration:

$$\int_{\mu_0}^{\mu} d\ln\mu' = \int_{x_0}^{x} f(x')\, dx'$$

$$\therefore \ln\frac{\mu}{\mu_0} = \int_{x_0}^{x} f(x')\, dx'$$

Indefinite integration:

$$\int^{\mu} d\ln\mu' = \int^{x} f(x')dx' + C$$

$$\therefore \ln\mu = \int^{x} f(x')\, dx' + C$$

$\therefore\ \mu = \mu_0 \exp \int_{x_0}^{x} f(x')\,dx'$ $\qquad\qquad$ $\therefore\ \mu = \exp\left(\int^{x} f(x')dx' + C \right)$

If you want, take $\mu_0 = 1$ $\qquad\qquad$ If you want, take $C = 0$

Let's work with $\boxed{\mu = \exp \int_{x_0}^{x} f(x')dx'}$

So our ode can be written as

$$(**) \qquad \frac{d(\mu y)}{dx} = \mu\, g\,, \quad \text{with } \mu(x) \text{ specified as above.}$$

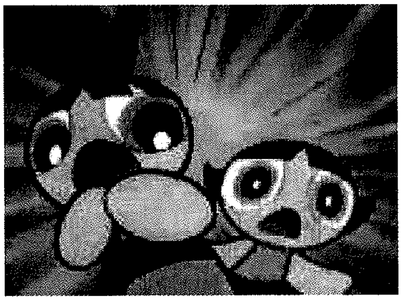
Integrate (**) -- it's essentially case I:

Use $y_0 = y(x_0)$, and remember we chose $\mu_0 = 1$

$$\therefore \quad \mu\, y - \mu(x_0)\, y(x_0) = \int_{x_0}^{x} \mu(x')\, g(x')\, dx'$$

$$\therefore \quad y = \frac{1}{\mu}\left[y_0 + \int_{x_0}^{x} \mu(x')\, g(x')\, dx' \right]$$

Or, write it out in it's full glory:



$$y = \exp - \int_{x_0}^{x} f(x')dx' \left[y_0 + \int_{x_0}^{x} g(x') \left(\exp \int_{x_0}^{x'} f(x'')dx'' \right) dx' \right]$$

There's no point in memorizing this equation. But, with practice it's easy to redo the derivation -- or solve particular problems. The key step is visualizing why the integrating factor is useful.

1st order <u>non-linear</u> odes are usually <u>more difficult to solve</u>. Can solve a few analytically (e.g., with separation of variables) but usually have to resort to numerical methods (e.g., Runge-Kutta).