Quick 'n' Dirty Review of Vector Operations
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<u>Scalars</u>

A quantity that has magnitude only is called a <u>scalar</u>. Examples: temperature T, pressure p, east-west speed u, gas constant R_d.

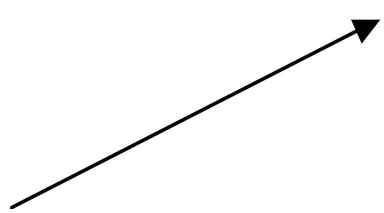
Scalars can be constant (like R_d) or functions of space and time (like T, p, u).

<u>Vectors</u>

A quantity that has magnitude and direction is a <u>vector</u>. e.g., acceleration $\vec{a}$, velocity $\vec{u}$, temp gradient ∇T, earth's angular velocity vector $\vec{\Omega}$.

Vectors can also be constant (like $\vec{\Omega}$) or functions of space and time (like $\vec{a}$, $\vec{u}$, ∇T).

A vector can be represented by an arrow:



Unit Vectors

A unit vector is a vector that has a magnitude of 1 (and is dimensionless). They are useful for expressing directions.

Consider an arbitrary vector $\vec{b}$. Its magnitude can be written as $b \equiv \left|\vec{b}\right|$. To get the unit vector in the direction of $\vec{b}$, divide $\vec{b}$ by its magnitude:

$$\hat{b} \equiv \frac{\vec{b}}{b}$$

The hat ^ means "unit vector". Multiplying this equation through by the magnitude b yields:

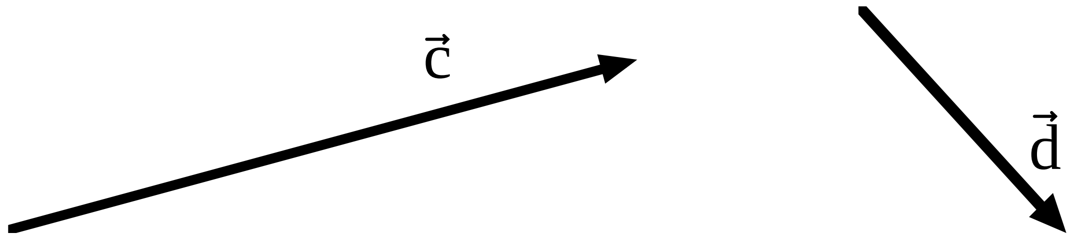
$$\boxed{\vec{b} = b\,\hat{b}}$$

This deceptively simple equation is extremely important. It means:

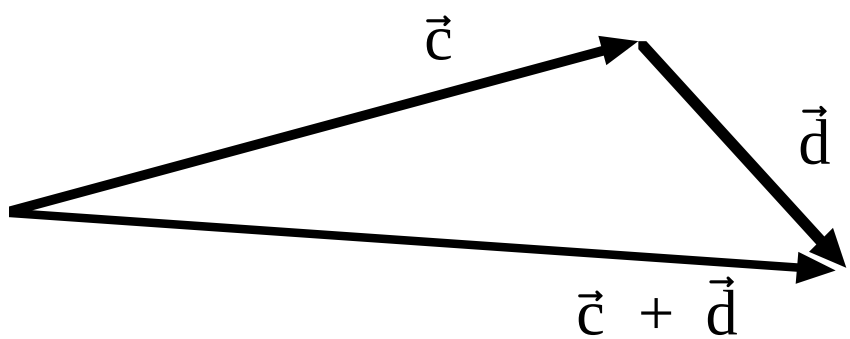
"Any vector can be written as the product of its own magnitude and the unit vector in its own direction"

Vector Addition

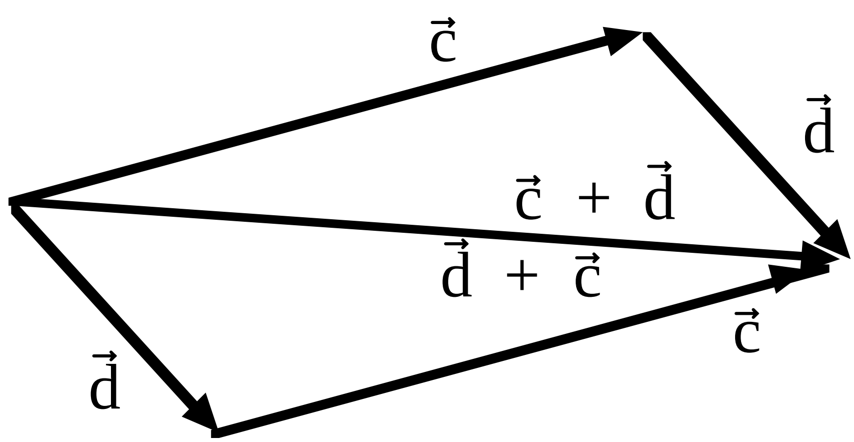
Consider 2 vectors, $\vec{c}$ and $\vec{d}$:



Adding $\vec{c}$ to $\vec{d}$ (tip of first to tail of the second) yields a new vector, $\vec{c} + \vec{d}$:



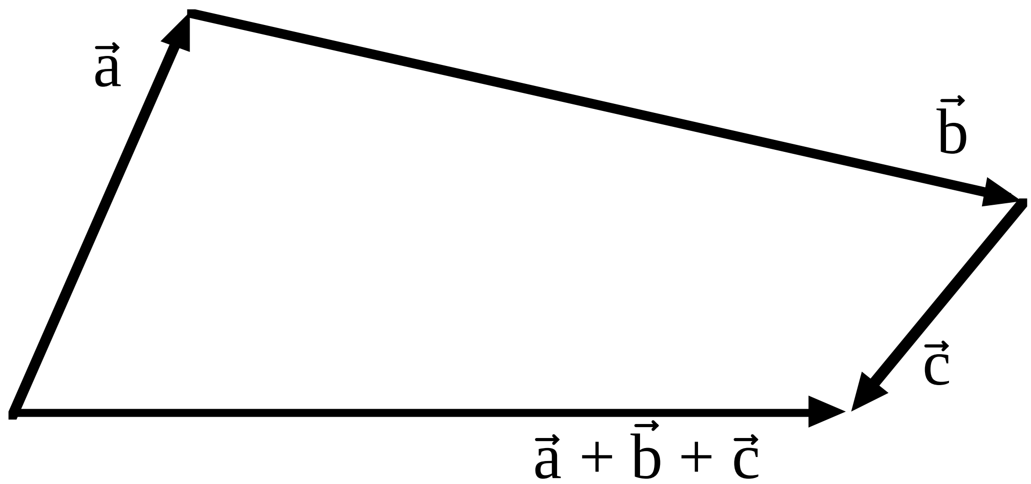
Get the same vector by adding $\vec{d}$ to $\vec{c}$:

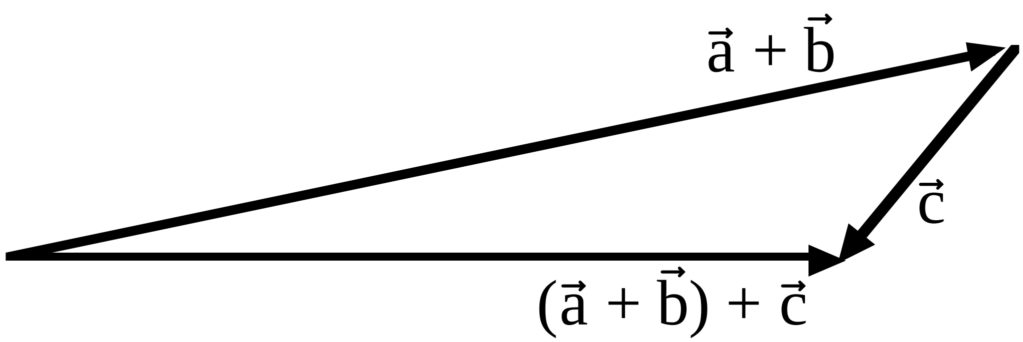


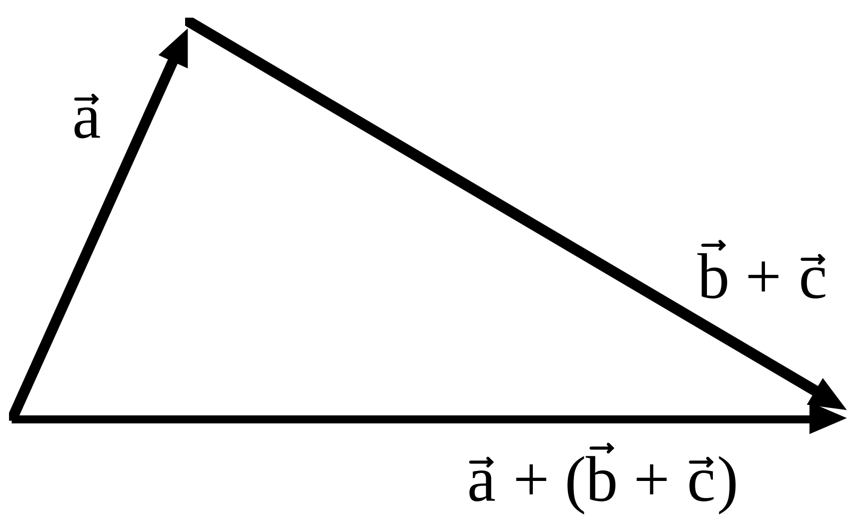
So, vector addition is <u>commutative</u>: $\vec{c} + \vec{d} = \vec{d} + \vec{c}$

Vector addition is also <u>associative</u>:

$$\vec{a} + \vec{b} + \vec{c} = \left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right).$$



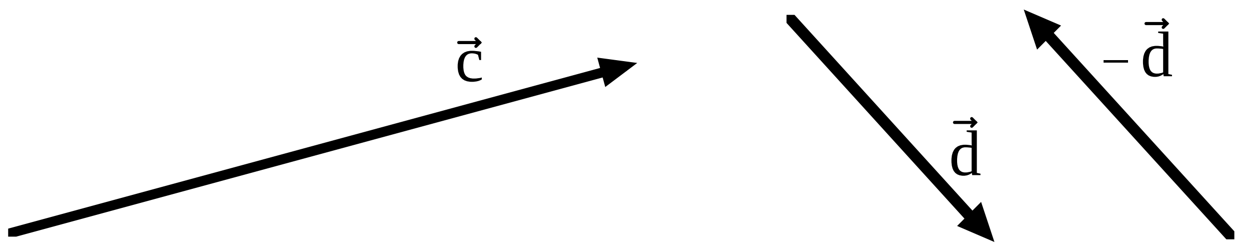


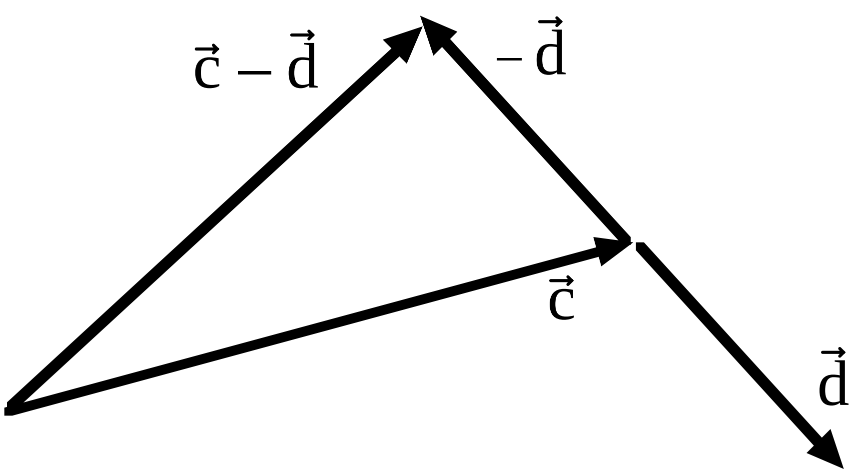


<u>Vector Subtraction</u>

Similar to vector addition: just add the <u>negative</u> of a vector.

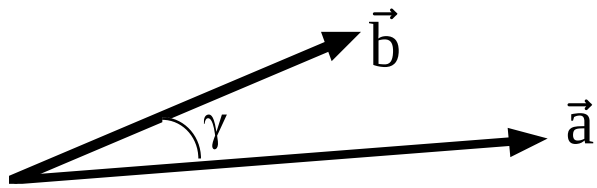
$$\vec{c} - \vec{d} = \vec{c} + \left(-\vec{d}\right)$$





Scalar product (dot product)

Consider two vectors $\vec{a}$ and $\vec{b}$. Let γ be the smallest angle between them:



The scalar (dot) product between $\vec{a}$ and $\vec{b}$ is:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\gamma .$$

It's a scalar! Can also write it as,

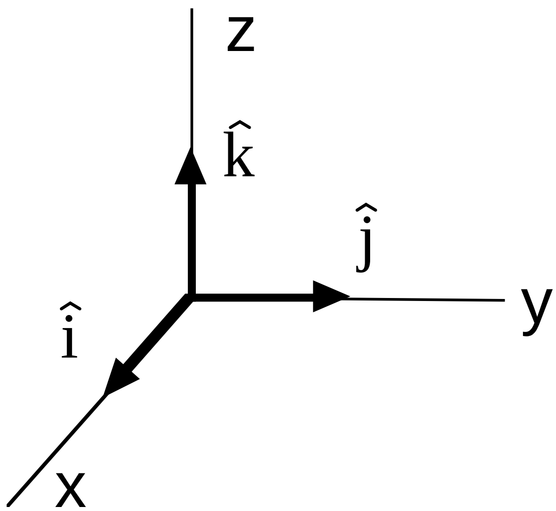
$$\vec{a} \cdot \vec{b} = ab\cos\gamma, \quad \text{where } a = |\vec{a}|, \; b = |\vec{b}|.$$

The dot product is commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$,
and distributive: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.

If $\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = 0$.
If $\vec{a} \cdot \vec{b} = 0$ then there are 3 possible scenarios:
(i) $\vec{a} = 0$, (ii) $\vec{b} = 0$, (iii) $\vec{a} \perp \vec{b}$.

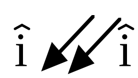
Unit vectors of Cartesian coord system

Consider a right-handed Cartesian coordinate system with unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$. $\hat{\imath}, \hat{\jmath}, \hat{k}$ are mutually $\perp$.

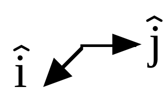


$$|\hat{\imath}| = 1, \qquad |\hat{\jmath}| = 1, \qquad |\hat{k}| = 1.$$

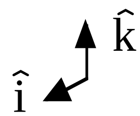
$$\hat{\imath} \cdot \hat{\imath} = |\hat{\imath}||\hat{\imath}| \cos 0 = 1$$



$$\hat{\imath} \cdot \hat{\jmath} = |\hat{\imath}||\hat{\jmath}| \cos 90 = 0$$



$$\hat{\imath} \cdot \hat{k} = |\hat{\imath}||\hat{k}| \cos 90 = 0$$

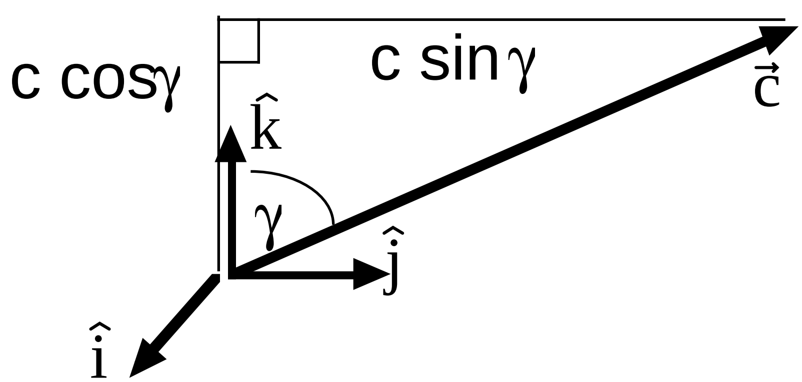


Similarly, $\hat{\jmath} \cdot \hat{\imath} = 0$, $\hat{\jmath} \cdot \hat{\jmath} = 1$, $\hat{\jmath} \cdot \hat{k} = 0$,
$\hat{k} \cdot \hat{\imath} = 0$, $\hat{k} \cdot \hat{\jmath} = 0$, $\hat{k} \cdot \hat{k} = 1$

Projection (component)

The projection (component) of a vector $\vec{c}$ in a particular direction is the dot product of $\vec{c}$ with the unit vector in that particular direction. It is the "amount" of $\vec{c}$ in that particular direction.

For example, the component of $\vec{c}$ in the vertical direction is $\vec{c} \cdot \hat{k}$. Call it c_z. So $c_z \equiv \vec{c} \cdot \hat{k}$ $= |\vec{c}||\hat{k}| \cos\gamma = c \cos\gamma$, where γ is the angle between $\vec{c}$ and $\hat{k}$, and $c \equiv |\vec{c}|$.



Similarly, the components of $\vec{c}$ in the $\hat{i}$ and $\hat{j}$ directions are: $c_x \equiv \vec{c} \cdot \hat{i}$, and $c_y \equiv \vec{c} \cdot \hat{j}$.

Can see that $\vec{c} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}$, i.e., $\vec{c}$ = sum of its components in the $\hat{i}$, $\hat{j}$, and $\hat{k}$ directions.

More on addition

Component of the sum of 2 vectors is equal to the sum of the components of the 2 vectors.

In other words, consider vectors $\vec{a}$ and $\vec{b}$, and their sum $\vec{c} \equiv \vec{a} + \vec{b}$. Write $\vec{a}$, $\vec{b}$ and $\vec{c}$ as,

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k},$$

$$\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k},$$

$$\vec{c} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}.$$

Then the sum of $\vec{a}$ and $\vec{b}$ is given by,

$$\vec{c} = \left(a_x + b_x\right)\hat{i} + \left(a_y + b_y\right)\hat{j} + \left(a_z + b_z\right)\hat{k}$$

So the i-component of $\vec{c}$ is: $c_x = a_x + b_x$.
So the j-component of $\vec{c}$ is: $c_y = a_y + b_y$.
So the k-component of $\vec{c}$ is: $c_z = a_z + b_z$.

More on the scalar (dot) product

With $\vec{a}$ and $\vec{b}$ written as, $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$, $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, the scalar product $\vec{a} \cdot \vec{b}$ becomes :

$$\vec{a} \cdot \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \cdot (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$$

$$\begin{aligned} = \; & a_x\hat{i} \cdot (b_x\hat{i} + b_y\hat{j} + b_z\hat{k}) + \\ & a_y\hat{j} \cdot (b_x\hat{i} + b_y\hat{j} + b_z\hat{k}) + \\ & a_z\hat{k} \cdot (b_x\hat{i} + b_y\hat{j} + b_z\hat{k}) \end{aligned}$$

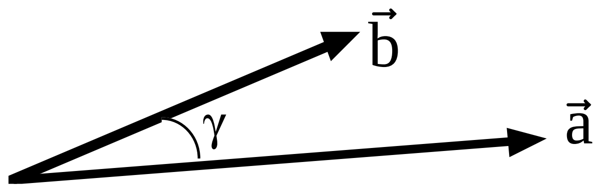
$$\begin{aligned} = \; & a_xb_x\hat{i} \cdot \hat{i} + a_xb_y\hat{i} \cdot \hat{j} + a_xb_z\hat{i} \cdot \hat{k} + \\ & a_yb_x\hat{j} \cdot \hat{i} + a_yb_y\hat{j} \cdot \hat{j} + a_yb_z\hat{j} \cdot \hat{k} + \\ & a_zb_x\hat{k} \cdot \hat{i} + a_zb_y\hat{k} \cdot \hat{j} + a_zb_z\hat{k} \cdot \hat{k} \end{aligned}$$

Since $\hat{i}, \hat{j}, \hat{k}$ are of <u>unit length</u> and are $\perp$ to each other, $\hat{i} \cdot \hat{i} = 1$, $\hat{i} \cdot \hat{j} = 0$, $\hat{i} \cdot \hat{k} = 0$, etc. So previous equation boils down to:

$$\vec{a} \cdot \vec{b} = a_xb_x + a_yb_y + a_zb_z$$

<u>Vector product (cross product)</u>

Consider two vectors $\vec{a}$ and $\vec{b}$. Let γ be the smallest angle between them:



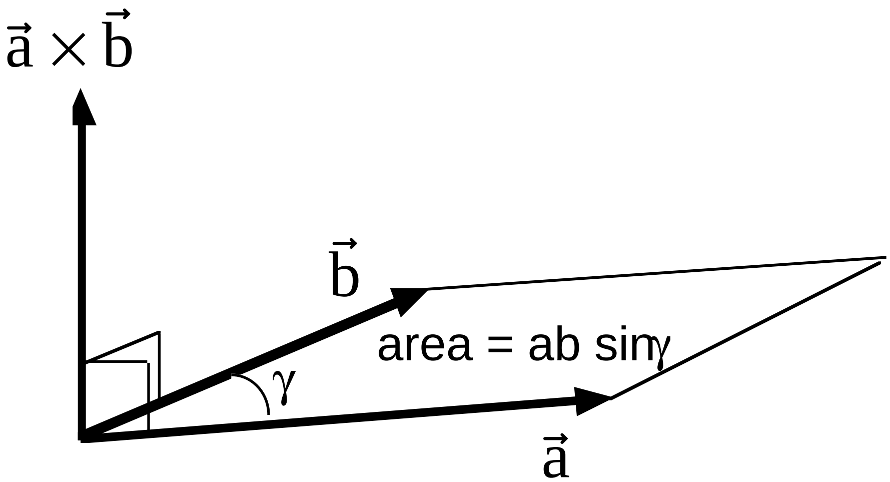
The <u>cross product</u> $\vec{a} \times \vec{b}$ is a <u>vector</u> with 3 properties:

(1) $\vec{a} \times \vec{b}$ is $\perp$ to <u>both</u> $\vec{a}$ and $\vec{b}$ (property 2 excludes one of two possible orientations).

(2) Right-hand rule for direction of $\vec{a} \times \vec{b}$: align fingers of your right hand with $\vec{a}$ then curl your fingers toward $\vec{b}$. Your thumb indicates direction of $\vec{a} \times \vec{b}$.

(3) <u>Magnitude</u> of $\vec{a} \times \vec{b}$ = area of trapezoid formed by $\vec{a}$ and $\vec{b}$: $\left| \vec{a} \times \vec{b} \right| = ab \sin \gamma$, where $a \equiv \left| \vec{a} \right|$, and $b \equiv \left| \vec{b} \right|$.

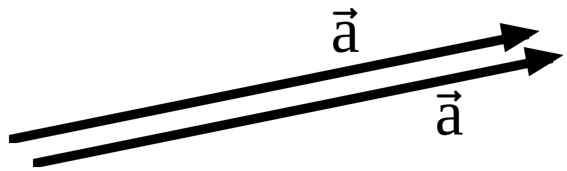
These three properties can be summarized on the following diagram:



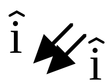
The angle γ between $\vec{a}$ and $\vec{b}$ that gives the largest magnitude of $\vec{a} \times \vec{b}$ is 90° (sin90 = 1, area is maximum). But when $\gamma = 90°$, the dot product $\vec{a} \cdot \vec{b}$ is 0 (since cos 90 = 0).

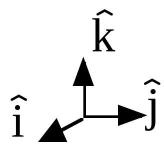
When $\vec{a}$ and $\vec{b}$ are parallel to each other ($\gamma = 0°$ or 180°, $\sin\gamma = 0$, area = 0) the cross product $\vec{a} \times \vec{b}$ is 0. But when $\gamma = 0°$ or 180°, the dot product $\vec{a} \cdot \vec{b}$ has its largest magnitude.

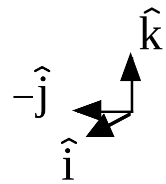
$$\vec{a} \times \vec{a} = 0$$



$$\hat{i} \times \hat{i} = 0, \qquad \hat{i} \times \hat{j} = \hat{k}, \qquad \hat{i} \times \hat{k} = -\hat{j}$$







Similarly,

$$\hat{j} \times \hat{i} = -\hat{k}, \qquad \hat{j} \times \hat{j} = 0, \qquad \hat{j} \times \hat{k} = \hat{i},$$

$$\hat{k} \times \hat{i} = \hat{j}, \qquad \hat{k} \times \hat{j} = -\hat{i}, \qquad \hat{k} \times \hat{k} = 0$$

It can be shown that $\vec{a} \times \vec{b}$ can be written as a determinant:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \hat{i}\left(a_y b_z - a_z b_y\right) + \hat{j}\left(a_z b_x - a_x b_z\right)$$

$$+ \hat{k}\left(a_x b_y - a_y b_x\right).$$

The cross product is <u>not</u> commutative because interchanging 2 rows of a determinant changes it's sign: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. This also follows from the right hand rule.

However, the cross product is <u>distributive</u>:
$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

<u>Scalar triple product</u>

Consider any 3 vectors, $\vec{a}$, $\vec{b}$ and $\vec{c}$, and form the expression,

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right).$$

This is a <u>scalar</u> known as the <u>scalar triple product</u>. (Why is it a scalar? $\vec{a}$ is a vector and $\vec{b} \times \vec{c}$ is a vector. The <u>dot product</u> between two vectors -- in this case $\vec{a}$ and $\vec{b} \times \vec{c}$ -- is a scalar.)

What does $\vec{a} \cdot (\vec{b} \times \vec{c})$ look like when expanded out?

Recall that $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$, or equivalently,

$$\vec{b} \times \vec{c} = \hat{i}\left(b_y c_z - b_z c_y\right) + \hat{j}\left(b_z c_x - b_x c_z\right)$$

$$+ \hat{k}\left(b_x c_y - b_y c_x\right).$$

Now take the dot product of $\vec{a}$ with $\vec{b} \times \vec{c}$:

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \left(\hat{i}\, a_x + \hat{j}\, a_y + \hat{k}\, a_z\right) \cdot \left(\vec{b} \times \vec{c}\right).$$

Plug in the expression for $\vec{b} \times \vec{c}$ and use fact that $\hat{i} \cdot \hat{i} = 1, \quad \hat{i} \cdot \hat{j} = 0, \quad \hat{i} \cdot \hat{k} = 0$, etc, to get,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = a_x(b_y c_z - b_z c_y) + a_y(b_z c_x - b_x c_z)$$

$$+ \; a_z(b_x c_y - b_y c_x).$$

This means that, $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$

If you interchange 2 rows once, the determinant changes sign. If you interchange 2 rows twice, the determinant stays the same. So:

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = -\vec{c} \cdot \left(\vec{b} \times \vec{a}\right) = -\vec{a} \cdot \left(\vec{c} \times \vec{b}\right)$$

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \vec{b} \cdot \left(\vec{c} \times \vec{a}\right) = \vec{c} \cdot \left(\vec{a} \times \vec{b}\right)$$

<u>Vector triple product</u>

The expression,

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right)$$

is a <u>vector</u> known as the <u>vector triple product</u>.

By examining the components of $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$ it can be shown that,

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b}\left(\vec{a} \cdot \vec{c}\right) - \vec{c}\left(\vec{a} \cdot \vec{b}\right).$$

The location of the parentheses does matter!
In general, $\vec{a} \times \left(\vec{b} \times \vec{c}\right) \neq \left(\vec{a} \times \vec{b}\right) \times \vec{c}$.
To see this for a particular case, let $\vec{b} = \vec{a}$ and compare $\vec{a} \times \left(\vec{a} \times \vec{c}\right)$ with $\left(\vec{a} \times \vec{a}\right) \times \vec{c}$.

$\vec{a} \times \left(\vec{a} \times \vec{c}\right) = \vec{a}\left(\vec{a} \cdot \vec{c}\right) - \vec{c}\left(\vec{a} \cdot \vec{a}\right)$, while
$\underbrace{\left(\vec{a} \times \vec{a}\right)}_{0} \times \vec{c} = 0$. Thus, they are not equal.

Differentiation rules

Suppose $\vec{a}$ and $\vec{b}$ are vectors and m is a scalar. Suppose $\vec{a}$, $\vec{b}$ and m are all functions of a scalar (say, time t). Then,

$$\frac{d}{dt}\left(\vec{a}+\vec{b}\right) = \frac{d\vec{a}}{dt} + \frac{d\vec{b}}{dt},$$

$$\frac{d}{dt}\left(m\vec{a}\right) = m\frac{d\vec{a}}{dt} + \frac{dm}{dt}\vec{a},$$

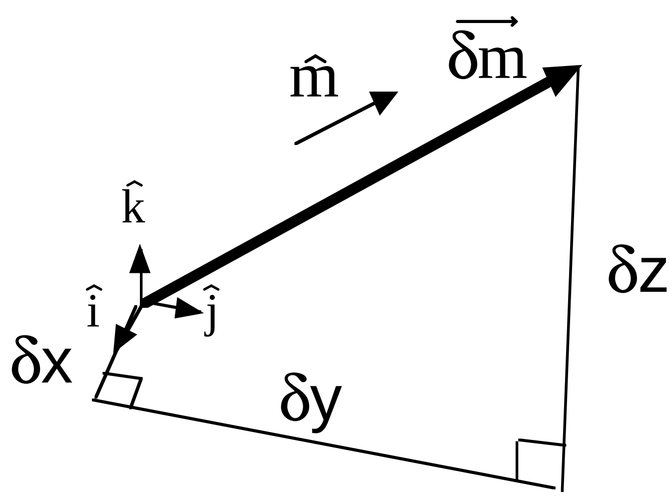
$$\frac{d}{dt}\left(\vec{a}\cdot\vec{b}\right) = \frac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\frac{d\vec{b}}{dt},$$

$$\frac{d}{dt}\left(\vec{a}\times\vec{b}\right) = \frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt}.$$

Directional derivative. Del operator ∇

The rate of change of temperature T in the $\hat{i}$ (east) direction is $\frac{\partial T}{\partial x}$. Similarly, the rate of change of temperature in the north and vertical directions are $\frac{\partial T}{\partial y}$, and $\frac{\partial T}{\partial z}$, respectively.

What about the rate of change of temperature in an arbitrary direction? Consider the direction specified by a unit vector $\hat{m}$. Consider a tiny vector element $\overrightarrow{\delta m}$ pointing in that direction.



$$\overrightarrow{\delta m} = \delta x\, \hat{i} + \delta y\, \hat{j} + \delta z\, \hat{k}, \qquad \delta m = \left| \overrightarrow{\delta m} \right|,$$

$$\hat{m} = \frac{\overrightarrow{\delta m}}{\delta m} = \frac{\delta x}{\delta m}\hat{i} + \frac{\delta y}{\delta m}\hat{j} + \frac{\delta z}{\delta m}\hat{k}.$$

$\overrightarrow{\delta m}$ extends from (x,y,z) to (x+δx, y+δy, z+δz). The temperature difference δT across this tiny distance $\overrightarrow{\delta m}$ is:

$$\delta T = T(x+\delta x, y+\delta y, z+\delta z) - T(x, y, z) .$$

Consider Taylor expansion of T about the starting point x, y, z:

$$T(x+\delta x, y+\delta y, z+\delta z) = T(x, y, z) +$$

$$+ \frac{\partial T}{\partial x}(x+\delta x - x) + \frac{\partial T}{\partial y}(y+\delta y - y)$$

$$+ \frac{\partial T}{\partial z}(z+\delta z - z) + \text{higher order terms}$$

$$\therefore \quad T(x+\delta x, y+\delta y, z+\delta z) - T(x, y, z) =$$

$$\frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z + \text{h.o.t}$$

$$\therefore \quad \delta T = \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z + \text{h.o.t.}$$

To get a rate of change of temperature in the direction of interest, divide δT by length δm.

$$\frac{\delta T}{\delta m} = \frac{\partial T}{\partial x}\frac{\delta x}{\delta m} + \frac{\partial T}{\partial y}\frac{\delta y}{\delta m} + \frac{\partial T}{\partial z}\frac{\delta z}{\delta m} + \frac{h.o.t}{\delta m}$$

Can rewrite this using dot product notation,

$$\frac{\delta T}{\delta m} = \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\delta x}{\delta m} \\ \frac{\delta y}{\delta m} \\ \frac{\delta z}{\delta m} \end{pmatrix} + \frac{h.o.t}{\delta m}$$

or, using $\hat{m} = \frac{\delta x}{\delta m}\,\hat{i} + \frac{\delta y}{\delta m}\,\hat{j} + \frac{\delta z}{\delta m}\,\hat{k}$:

$$\frac{\delta T}{\delta m} = \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix} \cdot \hat{m} + \frac{h.o.t}{\delta m}$$

In the limit of $\delta m \longrightarrow 0$, h.o.t $\longrightarrow 0$ and h.o.t/$\delta m \longrightarrow 0$ since h.o.t $\sim (\delta m)^2$. So get

$$\frac{\partial T}{\partial m} \equiv \lim_{\delta m \to 0} \frac{\delta T}{\delta m} = \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix} \cdot \hat{m}$$

Introduce the <u>del operator</u> ∇ defined by,

$$\nabla \equiv \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$$

∇ acting on a scalar f is ∇f. It's the <u>gradient</u> of f, a vector. Temperature gradient is given by,

$$\nabla T = \hat{i}\,\frac{\partial T}{\partial x} + \hat{j}\,\frac{\partial T}{\partial y} + \hat{k}\,\frac{\partial T}{\partial z} = \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix}$$

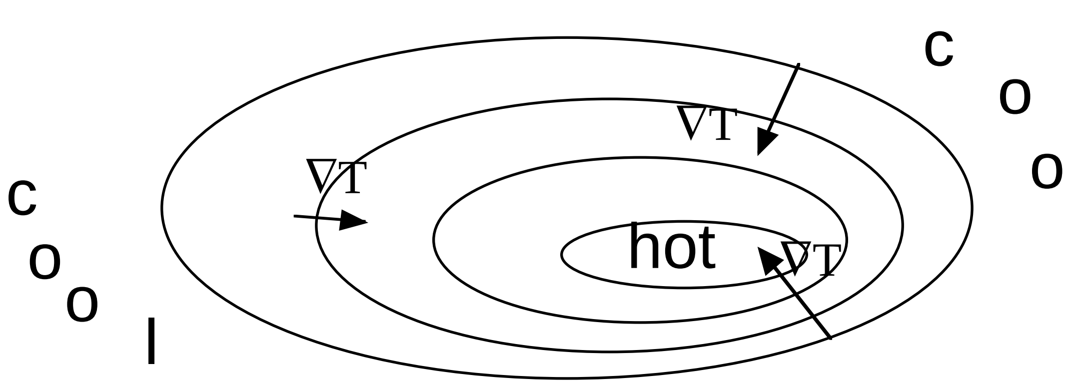
So the rate of change of T in the direction $\hat{m}$ (directional derivative in direction of $\hat{m}$) is given by,

$$\frac{\partial T}{\partial m} = \nabla T \cdot \hat{m} .$$

Important! If you consider a direction $\hat{m}$ to be perpendicular to ∇T then $\partial T/\partial m = 0$, i.e., there is no change in that direction. In other words, ∇T is $\perp$ to surfaces of constant T.

Also note that the largest positive value of $\partial T/\partial m$ is attained when $\hat{m}$ is in the direction of ∇T. So ∇T points in the direction of greatest change in T -- from lower to higher values of T.

These ideas are illustrated in this diagram:



Divergence

Take $\nabla \cdot$ (an arbitrary vector field $\vec{a}$):

$$\nabla \cdot \vec{a} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(\hat{i}a_x + \hat{j}a_y + \hat{k}a_z \right)$$

$$\begin{aligned} = \quad & \hat{i}\,\frac{\partial}{\partial x} \cdot \left(\hat{i}\,a_x + \hat{j}\,a_y + \hat{k}\,a_z \right) \\ + \; & \hat{j}\,\frac{\partial}{\partial y} \cdot \left(\hat{i}\,a_x + \hat{j}\,a_y + \hat{k}\,a_z \right) \\ + \; & \hat{k}\,\frac{\partial}{\partial z} \cdot \left(\hat{i}\,a_x + \hat{j}\,a_y + \hat{k}\,a_z \right) \end{aligned}$$

Expand out all the terms and recall that,
$\hat{i} \cdot \hat{i} = 1, \quad \hat{i} \cdot \hat{j} = 0, \quad \hat{i} \cdot \hat{k} = 0$, etc. Get:

$$\nabla \cdot \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} .$$

This is known as the divergence of $\vec{a}$. Note that $\vec{a}$ is a vector and ∇ is a vector operator. But $\nabla \cdot \vec{a}$ is a <u>scalar</u> (much as the dot product between 2 vectors is a scalar).

Curl

Consider an arbitrary vector field $\vec{a}$. See what happens when you take $\nabla \times \vec{a}$:

$$\nabla \times \vec{a} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \times \left(\hat{i} a_x + \hat{j} a_y + \hat{k} a_z \right)$$

$$\begin{aligned} &= \hat{i}\frac{\partial}{\partial x} \times \left(\hat{i}\, a_x + \hat{j}\, a_y + \hat{k}\, a_z \right) \\ &\quad + \hat{j}\frac{\partial}{\partial y} \times \left(\hat{i}\, a_x + \hat{j}\, a_y + \hat{k}\, a_z \right) \\ &\quad + \hat{k}\frac{\partial}{\partial z} \times \left(\hat{i}\, a_x + \hat{j}\, a_y + \hat{k}\, a_z \right) \end{aligned}$$

Expand out all the terms and recall that,
$\hat{i} \times \hat{i} = 0$, $\hat{i} \times \hat{j} = \hat{k}$, $\hat{i} \times \hat{k} = -\hat{j}$, etc. Get:

$$\nabla \times \vec{a} = \hat{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) + \hat{j}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \right) + \hat{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right).$$

$\nabla \times \vec{a}$ is known as the <u>curl</u> of $\vec{a}$. It is a <u>vector</u>!

Don't want to memorize that nasty formula for $\nabla \times \vec{a}$? No problem! The curl can also be written as a determinant:

$$\nabla \times \vec{a} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}a_x + \hat{j}a_y + \hat{k}a_z\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \hat{j}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \hat{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right).$$

Same result as before.

<u>Differentiation formulas involving</u> ∇

If $\vec{a}$ and $\vec{b}$ are vectors and f is a scalar then,

$$\nabla \cdot \left(\vec{a} + \vec{b}\right) = \nabla \cdot \vec{a} + \nabla \cdot \vec{b} \,,$$

$$\nabla \times \left(\vec{a} + \vec{b}\right) = \nabla \times \vec{a} + \nabla \times \vec{b} \,,$$

$$\nabla \cdot \left(f\vec{a}\right) = f\,(\nabla \cdot \vec{a}) + (\nabla f) \cdot \vec{a} \,,$$

$$\nabla \times \left(f\vec{a}\right) = f\,(\nabla \times \vec{a}) + (\nabla f) \times \vec{a} \,,$$

$$\nabla \cdot \left(\vec{a} \times \vec{b}\right) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b}) \,,$$

$$\nabla \times \left(\vec{a} \times \vec{b}\right) = \vec{a}\,(\nabla \cdot \vec{b}) + (\vec{b} \cdot \nabla)\,\vec{a}$$

$$- \vec{b}\,(\nabla \cdot \vec{a}) - (\vec{a} \cdot \nabla)\,\vec{b} \,,$$

$$\nabla \left(\vec{a} \cdot \vec{b}\right) = (\vec{a} \cdot \nabla)\,\vec{b} + (\vec{b} \cdot \nabla)\,\vec{a}$$

$$+ \vec{a} \times (\nabla \times \vec{b}) + \vec{b} \times (\nabla \times \vec{a}) \,.$$

Further Reading on Vector Analysis

Baxandall, P., and Liebeck, H., 1986: *Vector Calculus*. Oxford University Press, 550 pp.
A detailed, rigorous and painful account of vector calculus. Much more mathematical than the other texts on this list. Not for the squeamish.

Hay, G. E., 1953: *Vector and Tensor Analysis*. Dover, 193 pp.
A good book: well-written, clear, complete, and relatively short. The basics on vector analysis are covered in just 3 short chapters: I, IV and V. The remainder of the book covers applications of vector analysis and a short section on tensors. It's a Dover paperback = cheap!

Kreyszig, E. K., 1993: *Advanced Engineering Mathematics*, 7th ed. Wiley, ~ 1000 pp.
It's a good reference for many aspects of applied math (advanced calculus, differential equations, complex analysis, vector analysis, etc). It is well-written and is sufficiently complete for your vector analysis needs. Can buy it online for next to nothing.

Schey, H. M., 1992: *Div, Grad, Curl and All That*, 2nd ed. W. W. Norton, 163 pp.
I prefer the Hay book to this one but maybe you'll like this one better. It's well-written and short, but not quite as complete as the Hay book.