METR 5113, Advanced Atmospheric Dynamics I
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Mass Conservation

Consider an infinitesimal fluid parcel of density ρ and volume δV. It's mass is unchanged no matter how it moves or deforms,

$$\rho\,\delta V = \text{const (for a parcel)}$$

So as $\delta V\uparrow$, $\rho\downarrow$

$$\therefore \quad \frac{D(\rho\,\delta V)}{Dt} = 0$$

$$\rho\frac{D\delta V}{Dt} + \delta V\frac{D\rho}{Dt} = 0 \qquad \div \text{ by } \rho\,\delta V$$

$$\text{(A)} \quad \frac{1}{\delta V}\frac{D\delta V}{Dt} + \frac{1}{\rho}\frac{D\rho}{Dt} = 0$$

These are all equivalent Lagrangian expressions of mass cons^n.

To get a hybrid Lagrangian/Eulerian form of mass cons^n, apply $\frac{1}{\delta V}\frac{D\delta V}{Dt} = \frac{\partial u_i}{\partial x_i}$ in (A) to get:

$$(*) \quad \boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial u_i}{\partial x_i} = 0} \quad \text{or} \quad \boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{u} = 0}$$

Eqn (*) is called the Mass Conservation eq^n. [Also referred to as "mass continuity eq^n" but that's a bad name for it -- mass is no more or less continuous than any other variable.]

Get a <u>purely Eulerian form</u> of mass conservation, by expressing $D\rho/Dt$ in (*) in Eulerian form.

$$(**) \qquad \boxed{\frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0}$$

Combine last 2 terms to get "flux form" of mass $cons^n$ eq^n:

$$(***) \qquad \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0} \quad \text{or} \quad \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0}$$

Eq^{ns} (**) and (***) are equivalent, purely Eulerian forms of mass $cons^n$.

All of the above eq^{ns} (Lag, Eulerian, or hybrid) are <u>exact</u> forms of mass $cons^n$ -- no approximations made.

<u>Some approximate forms of mass $cons^n$</u>

Let $\rho_0(z)$ be horiz average of ρ. Define perturbation density, ρ':

$$\underset{\rho'(x,y,z,t)}{\underset{\downarrow}{\rho'}} \equiv \underset{\rho(x,y,z,t)}{\underset{\downarrow}{\rho}} - \underset{\rho_0(z)}{\underset{\downarrow}{\rho_0}} \qquad [\rho = \rho_0 + \rho']$$

Substitute $\rho = \rho_0 + \rho'$ into (**),

$$\frac{\partial \rho'}{\partial t} + \underset{0}{\boxed{\frac{\partial \rho_0}{\partial t}}} + \vec{u}\cdot\nabla\rho' + \underset{w\, d\rho_0/dz}{\boxed{\vec{u}\cdot\nabla\rho_0}} + (\rho' + \rho_0)\nabla\cdot\vec{u} = 0$$

$$\frac{\partial \rho'}{\partial t} + \vec{u} \cdot \nabla \rho' + w \frac{d\rho_0}{dz} + (\rho' + \rho_0) \nabla \cdot \vec{u} = 0$$

Typically ρ' is much smaller than ρ_0 ($\rho' \sim 1$ - 3% of ρ_0).

A good approx to mass cons eq^n is obtained by neglecting ρ':

$$w \frac{d\rho_0}{dz} + \rho_0 \nabla \cdot \vec{u} = 0$$

÷ by ρ_0, get:

$$\boxed{\nabla \cdot \vec{u} + w \frac{d\ln\rho_0}{dz} = 0}$$

Can also write it as,

$$\boxed{\nabla \cdot (\rho_0 \vec{u}) = 0}$$

These approx forms of mass cons are known as the <u>anelastic</u> mass conservation eq^n or <u>deep convection mass cons eq^n</u>

- Commonly used in synoptic and mesoscale models + theories.

- Sound waves filtered out.

<u>Incompressibility cond^n</u>

Neglect <u>all changes</u> in density in mass cons eq^n. Get:

$$\boxed{\nabla \cdot \vec{u} = 0} \qquad \text{Incompressibility condition.}$$

$$\therefore \frac{1}{\delta V}\frac{D\delta V}{Dt} = 0 \quad \therefore \delta V = \text{const} \quad \therefore \underline{\text{volume is conserved}}$$

- appropriate approximation to mass cons eq^n for liquids (oceans, lakes) and for atmosphere if vertical motion is "small", i.e., if vertical scale of motion is much less than vertical scale over which ρ changes appreciably (e-folding height ~ 10km).

- incompressibility $cond^n$ also known as shallow convection mass $cons^n$ eq^n. Used in mesoscale modelling + theories + turbulence modelling.

Two-dimensional incompressible flows

Consider hypothetical flow that is two-dimensional, $u = u(x, y)$, $v = v(x, y)$, $w = 0$ and incompressible, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$.
Since w=0 everywhere this becomes $\partial u/\partial x + \partial v/\partial y = 0$, which is satisfied for u and v that satisfy:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x},$$

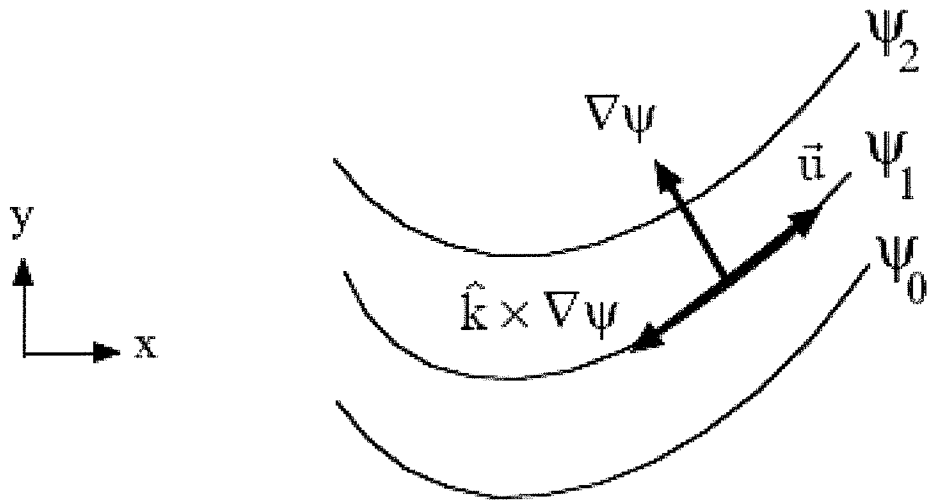
where ψ is a streamfunction. Proof:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \boxed{\frac{\partial^2 \psi}{\partial y \partial x}} = 0$$

↓

interchange order of differentiation, get $\frac{\partial^2 \psi}{\partial x\, \partial y}$, which cancels with first term

Can write $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$ in vector form as: $\vec{u} = -\hat{k} \times \nabla\psi$.
Graphically, this means:



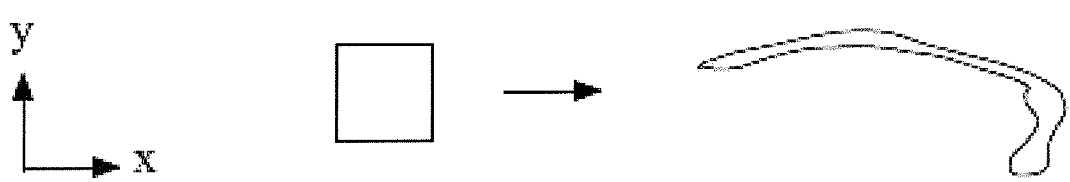
So, in <u>2D incomp flows, curves of constant ψ are streamlines!</u>

Note: Since $\frac{1}{\delta x}\frac{D\delta x}{Dt} = \frac{\partial u}{\partial x}$ and $\frac{1}{\delta y}\frac{D\delta y}{Dt} = \frac{\partial v}{\partial y}$,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{\delta x}\frac{D\delta x}{Dt} + \frac{1}{\delta y}\frac{D\delta y}{Dt} = \frac{1}{\delta x \delta y}\frac{D(\delta x \delta y)}{Dt} = \frac{1}{\delta A}\frac{D\delta A}{Dt},$$

where $\delta A \equiv \delta x \delta y$ is horiz cross-sectional area of parcel.

So if $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ then $\frac{1}{\delta A}\frac{D\delta A}{Dt} = 0$ so then $\delta A = \text{const.}$
∴ <u>Horiz area of parcels doesn't change in 2D incomp flows.</u>



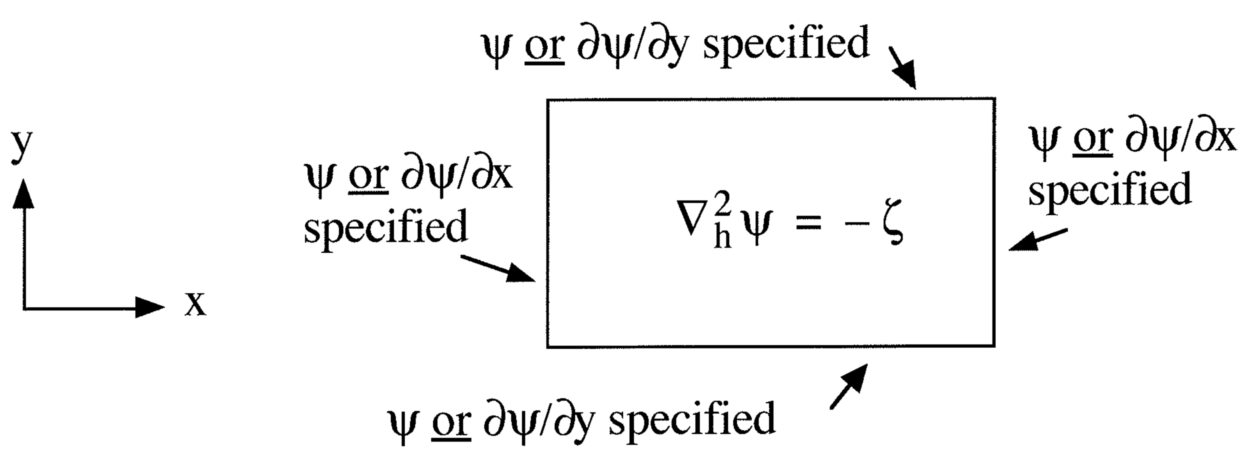
For 2-D incomp flows, $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$, and vertical vorticity is:

$$\zeta \equiv \hat{k} \cdot (\nabla \times \vec{u}) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = -\nabla_h^2 \psi$$

Suppose we know ζ everywhere and want to find u, v assuming flow is 2-D and incompressible (as in early days of NWP). Get ψ by solving

$$\nabla_h^2 \psi = -\zeta ,$$ 2nd order linear elliptic pde (Poisson eqn),

where the right hand side is a known function of x and y. Need boundary condns to solve this eqn. Can specifiy ψ on bdry (Dirichlet condn) or normal derivative of ψ on bdry (Neumann condn). Can mix and match from one bdry point to another -- but can't put both D and N condns on same point.



Solve $\nabla_h^2 \psi = -\zeta$ for ψ, (use SOR or other numerical technique) then differentiate it to get u (= $\partial \psi / \partial y$) and v (= $-\partial \psi / \partial x$).