METR 5113, Advanced Atmospheric Dynamics I
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- **4 handouts:** answers to prob set 2, vort and div in cylindrical coords, Laplacian in cylindrical coords, Stokes streamfunction and reg streamfunction

2-D Incompressible flows (continued)

Consider a streamline drawn in a 2-D incomp flow.

dx, dy are increments along a streamline.

So differential eqn for this streamline is: $\frac{dx}{u} = \frac{dy}{v}$ along streamline. Plug in expressions for u,v in terms of ψ

$$\frac{dx}{\frac{\partial\psi}{\partial y}} = \frac{dy}{-\frac{\partial\psi}{\partial x}} \quad \text{along streamline}$$

Cross multiply and rearrange to get:

$$\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = 0 \quad \text{along streamline}$$

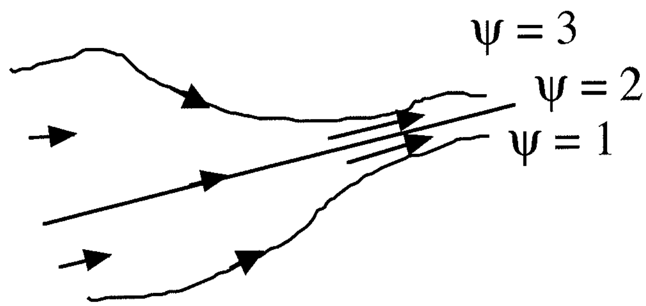
[_____ $d\psi$ _____] Left hand side of this eqn is $d\psi$ expanded with chain rule

$$\therefore \; d\psi = 0 \qquad \text{along streamline}$$

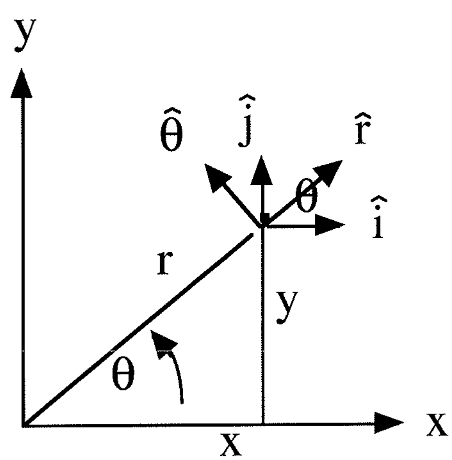
$$\therefore \; \psi = \text{const} \quad \text{(along streamline)}$$

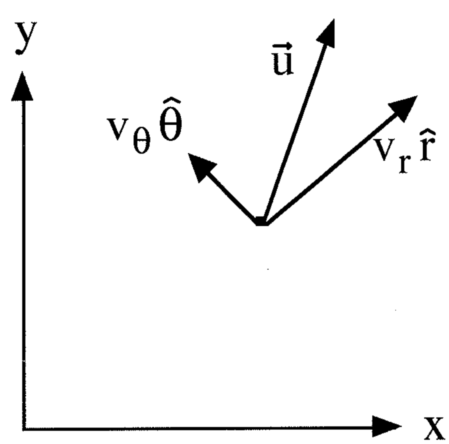
Streamlines always exist (2-D/3-D, compressible/incomp flows) but a streamfunction ψ defined by $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$ only exists for 2-D incomp flows.

In 2D incomp flows, winds are tangent to lines of constant ψ and large gradients of ψ imply strong winds:



Sometimes it's convenient to work in cylindrical polar coords. We will relate <u>radial</u> and <u>azimuthal</u> velocity components to ψ.





$\hat{r}$ is unit vector in dir^n of r increasing
$\hat{\theta}$ is unit vector in dir^n of θ increasing
$v_r \equiv \hat{r} \cdot \vec{u}$ radial velocity comp
$v_\theta \equiv \hat{\theta} \cdot \vec{u}$ azimuthal (tangential) velocity comp
$x = r\cos\theta$, $y = r\sin\theta$
$\therefore\ r^2 = x^2 + y^2$
$\tan\theta = y/x \qquad \therefore\ \theta = \tan^{-1}(y/x)$

$\psi(x, y) = \psi[x(r, \theta), y(r, \theta)]$

Can relate radial and azimuthal derivs of ψ to azimuthal and radial velocity comps (analogous to $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$):

Calculate radial and azimuthal derivatives of ψ:

$$\frac{\partial\psi}{\partial r} = \underbrace{\left.\frac{\partial\psi}{\partial r}\right|_{\theta}}_{\text{holding } \theta \text{ const}} = \left.\frac{\partial\psi}{\partial x}\right|_{y}\left.\frac{\partial x}{\partial r}\right|_{\theta} + \left.\frac{\partial\psi}{\partial y}\right|_{x}\left.\frac{\partial y}{\partial r}\right|_{\theta} \qquad \text{(chain rule)}$$

$$= -v\cos\theta + u\sin\theta$$

$$\frac{\partial\psi}{\partial\theta} = \text{(show work)} = v\,r\sin\theta + u\,r\cos\theta$$

Obtain similar-looking expressions for v_r and v_θ :

$$v_r = \hat{r}\cdot\vec{u} = \hat{r}\cdot(u\,\hat{i} + v\,\hat{j}) = u\cos\theta + v\cos(90-\theta)$$

$$= u\cos\theta + v\,(\underbrace{\cos 90}_{0}\cos\theta + \underbrace{\sin 90}_{1}\sin\theta)$$

$$= u\cos\theta + v\sin\theta$$

$$v_\theta = \hat{\theta}\cdot\vec{u} = \text{(show work)} = -u\sin\theta + v\cos\theta$$

$\therefore$ can see that: $\boxed{v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}}$ $\qquad$ $\boxed{v_\theta = -\frac{\partial\psi}{\partial r}}$

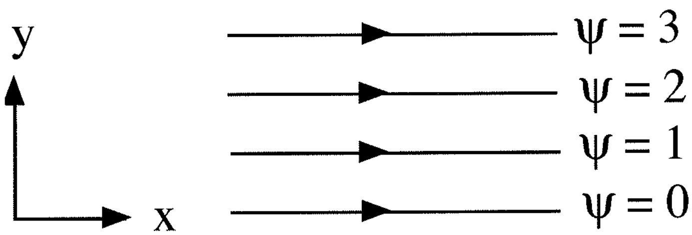
In cyl coords, vert vort is: $\zeta = \frac{1}{r}\frac{\partial(r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}$ (see handout)

Examples of simple 2-D incomp flows

(1) $\psi = U\,y$ where U is const

$$\therefore\ u = \frac{\partial \psi}{\partial y} = U$$

$$v = -\frac{\partial \psi}{\partial x} = 0$$



$\therefore\ \vec{u} = U\,\hat{i}$ const, unidirectional flow

(2) $\psi = U\,y + D$ where U and D are const.

Get same flow as above, $u = \frac{\partial \psi}{\partial y} = U, \quad v = -\frac{\partial \psi}{\partial x} = 0$

(3) $\psi = -\frac{K}{2} r^2$ where K is a const.

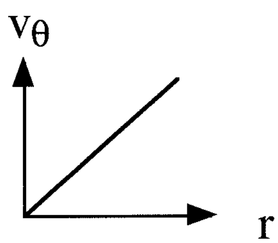
$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

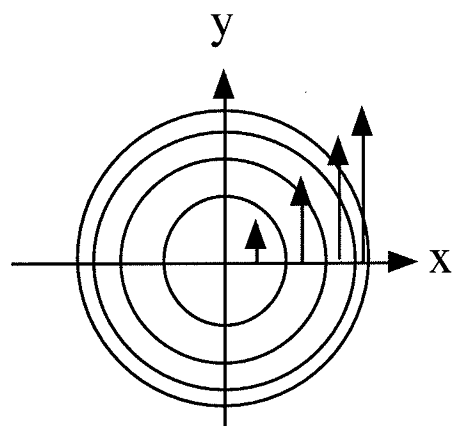
$$v_\theta = -\frac{\partial \psi}{\partial r} = -\left(-\frac{K}{2}\,2\,r\right) = Kr$$

$\therefore\ \vec{u} = Kr\,\hat{\theta}$

ang velocity $\Omega \equiv \frac{v_\theta}{r} = K$ so ang velocity is const.

Flow is a <u>solid body vortex.</u> (like planet or a record player)





vert vorticity: $\zeta = \frac{1}{r}\frac{\partial(rv_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = \frac{1}{r}\frac{\partial(Kr^2)}{\partial r} = 2K$

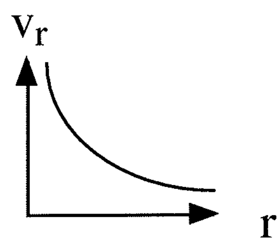
(4) $\psi = \frac{Q}{2\pi}\theta$ where Q is const.

$\therefore\ v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi r}, \qquad v_\theta = -\frac{\partial \psi}{\partial r} = 0$

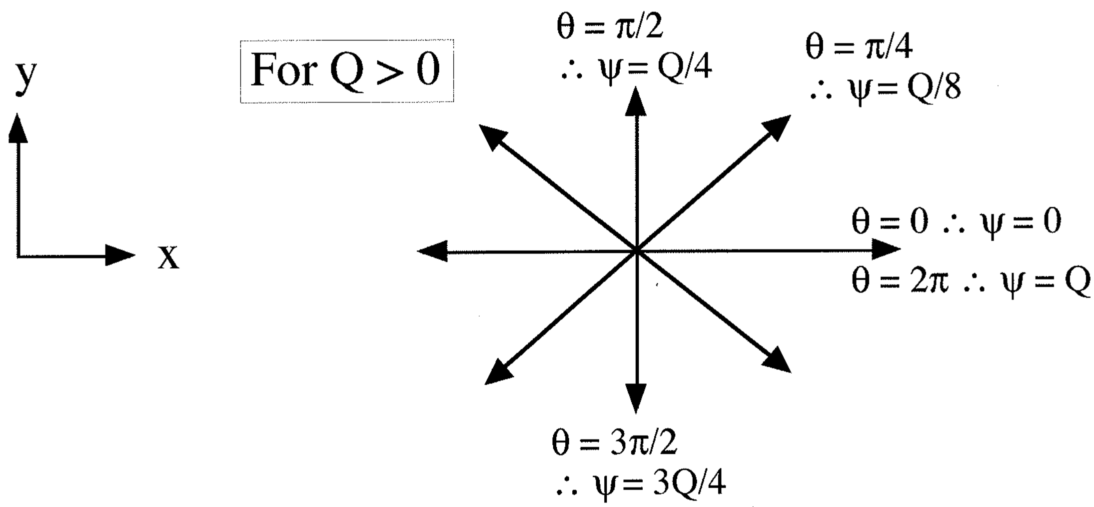
radial flow only (inflow if $Q < 0$, outflow if $Q > 0$)

$\therefore\ \vec{u} = \frac{Q}{2\pi r}\hat{r}$

The closer to the origin, the faster the radial velocity. $v_r \to \infty$ as $r \to 0$.



In horiz plane it looks like:



A <u>line source</u> for $Q > 0$, a <u>line sink</u> for $Q < 0$ (arrows reversed on above diagram). The "line" is the z-axis.

Can show that $\zeta = 0$ for this case.

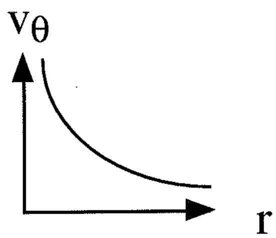
Note: ψ is multivalued (on positive x axis $\psi = 0$ and Q. Be careful - do not differentiate ψ across positive x axis).

(5) vr-vortex

$$\psi = -\frac{\Gamma}{2\pi}\ln r$$

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0 \quad \text{no radial wind}$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$



v_θ is infinite at $r = 0$.

Angular velocity $\frac{v_\theta}{r}$ is $\frac{\Gamma}{2\pi r^2}$ -- it decreases with radius.

Angular momentum $v_\theta r$ is $\frac{\Gamma}{2\pi}$ -- a constant

Vert vorticity $\zeta = \frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}\frac{\Gamma}{2\pi} = 0$ [except at $r=0$; $\div$ by 0 is illegal]. Using Stokes th^m, can show $\zeta = \infty$ at $r=0$ [try it]. So vr-vortex is an irrotational vortex (except at origin).

(6) Rankine Vortex: Solid body vortex in inner region "patched" to vr vortex in outer region.