METR 5113, Advanced Atmospheric Dynamics I
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Reading: Kundu chapter on Conservation Laws

1 handout: Leibnitz rule (Kundu's figure)

Helmholtz Theorem: a general flow decomposition

Real flows can be 3D, and comp or incomp. Horiz velocity is $\vec{V}_h \equiv u\,\hat{i} + v\,\hat{j}$. Since flow is not necessarily 2D and incomp, can't say $\vec{V}_h = -\hat{k}\times\nabla\psi$. However, Helmholtz theorem says there always exist scalar functions χ (velocity potential) and ψ (streamfunction) such that:

$$(*) \qquad \boxed{\vec{V}_h = \nabla_h\chi - \hat{k}\times\nabla_h\psi} . \qquad\qquad [\nabla_h \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}]$$

It's a partition of the horiz wind into a rotational part (associated w/ ψ) and a horizontally divergent part (associated w/ χ).

In components, (*) is: $\quad u = \dfrac{\partial\chi}{\partial x} + \dfrac{\partial\psi}{\partial y}, \quad v = \dfrac{\partial\chi}{\partial y} - \dfrac{\partial\psi}{\partial x}$

Consider vertical vorticity:

interchange order of dif get cancellation

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2\chi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x^2} - \left(\boxed{\frac{\partial^2\chi}{\partial y\partial x}} + \frac{\partial^2\psi}{\partial y^2}\right) = -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}$$

$$\therefore \quad \zeta = -\nabla_h^2\psi$$

Consider horiz divergence:

interchange order of dif, get cancellation

$$\delta_h \equiv \nabla_h \cdot \vec{V}_h = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} + \left(\frac{\partial^2 \chi}{\partial y^2} - \boxed{\frac{\partial^2 \psi}{\partial y \partial x}} \right)$$

$$= \frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2}$$

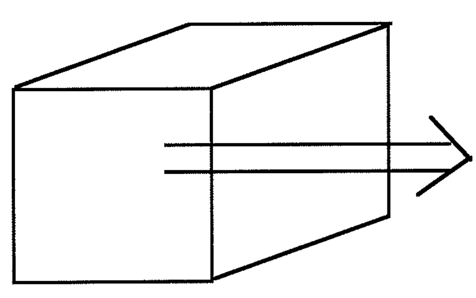
$$\therefore \quad \delta_h = \nabla_h^2 \chi$$

If you know δ_h and ζ, solve $\nabla_h^2 \psi = -\zeta$ for ψ and $\nabla_h^2 \chi = \delta$ for χ, then get wind from (*). Details in Lynch (Mon. Wea. Rev. 1988). This decomposition is used in theoretical work, NWP models and data assimilation (e.g., MM5 and WRF models).

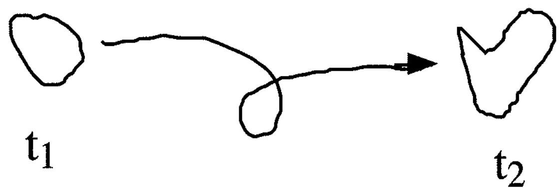
Conservation Laws (Ch4 Kundu)

We will consider 2 types of volumes:

(i) fixed volume e.g., imaginary box that doesn't move. Flow moves through it:

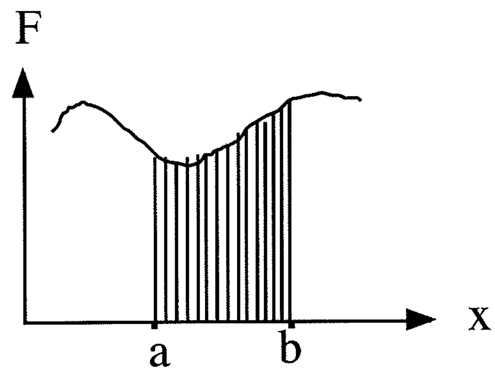


(ii) parcel volume (material volume) A volume that moves with the local velocity field:



[3rd type of vol: moving but not w/ flow, e.g. following a geometrical pattern like a wave or a mesocyclone]

Let F(x,t) be a f^n of x and time t. Total amount of F in a time-dependent domain btw x = a(t) and x = b(t) is $\int_{a(t)}^{b(t)} F(x,t)\,dx$.



This total amount of F can increase with time if F increases btw a and b or if domain becomes bigger (b increases and/or a decreases). This is quanitifed in 1D Leibnitz rule:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)\,dx = \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t}\,dx + F(b,t)\,\frac{db}{dt} - F(a,t)\,\frac{da}{dt}$$

- look at handout on Leibnitz rule (Kundu fig.)

In 3-D, Leibnitz rule becomes:

$$\frac{d}{dt}\int_{V(t)} F(\vec{x},t)\,dV = \int_{V(t)} \frac{\partial F}{\partial t}\,dV + \int_{A(t)} F\,\vec{u}_A \cdot \hat{n}\,dA$$

where $\vec{u}_A$ is velocity of the area element dA. A(t) is the sfc bounding the volume V(t). $\hat{n}$ is unit outward normal vector. If $\vec{u}_A \cdot \hat{n} > 0$ bdry is moving <u>outward</u>.

For a <u>fixed volume</u>, $\vec{u}_A = 0$,

$$\therefore \quad \frac{d}{dt}\int_V F(\vec{x}, t)\, dV = \int_V \frac{\partial F}{\partial t}\, dV$$

For a <u>material volume</u>, $\vec{u}_A = \vec{u}$, local fluid velocity on bdry (also d/dt = D/Dt total deriv)

$$\therefore \quad \frac{D}{Dt}\int_{V(t)} F(\vec{x}, t) dV = \int_{V(t)} \frac{\partial F}{\partial t} dV + \boxed{\int_{A(t)} F\,\vec{u}\cdot\hat{n}\, dA}$$

↓ use div theorem

$$\int_{V(t)} \nabla\cdot(F\,\vec{u}) dV$$

↓ use product rule

$$\boxed{F\,\nabla\cdot\vec{u} + \vec{u}\cdot\nabla F}$$

$$\therefore \quad \frac{D}{Dt}\int_{V(t)} F(\vec{x}, t)\, dV = \int_{V(t)} \left[\frac{DF}{Dt} + F\,\nabla\cdot\vec{u}\right] dV$$

To study how ρF changes in a volume (where $\rho = \rho(\vec{x}, t)$), replace F by ρF in prev results:

For a fixed volume,

$$\frac{d}{dt}\int_V \rho F(\vec{x},t)\,dV = \int_V \frac{\partial(\rho F)}{\partial t}\,dV$$

For a <u>material</u> volume:

$$\frac{D}{Dt}\int_{V(t)} \rho\, F\, dV = \int_{V(t)} \left[\boxed{\frac{D(\rho F)}{Dt}} + \rho F\, \nabla \cdot \vec{u} \right] dV$$

$$\downarrow$$

$$\rho\frac{DF}{Dt} + F\frac{D\rho}{Dt}$$

$$= \int_{V(t)} \left[\rho \frac{DF}{Dt} + F \boxed{\left(\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} \right)} \right] dV$$

$$\downarrow$$

0 from mass cons

$$\therefore \quad \boxed{\frac{D}{Dt}\int_{V(t)} \rho F\, dV = \int_{V(t)} \rho \frac{DF}{Dt}\, dV}$$

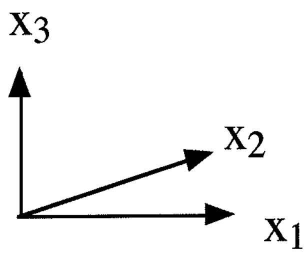
<u>Forces</u>

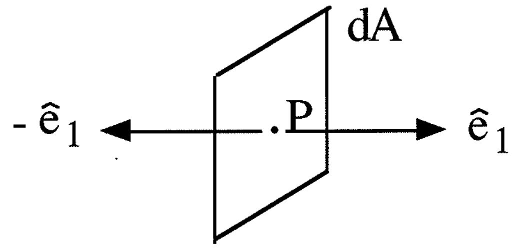
2 types of forces on an air parcel:

(1) <u>Body forces</u>. Proportional to volume (mass) of parcel. e.g., gravity force: $\int \rho\vec{g}\,dV$. Due to "action at a distance".

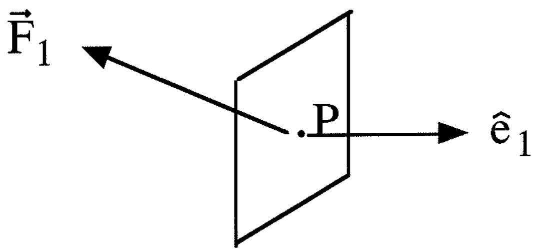
(2) <u>Surface forces</u> (<u>stress forces</u>). Forces exerted on sfc of a blob by fluid outside blob. Force due to <u>contact</u> of element w/ surroundings. Depends on location + orientation of element.

Now lets carefully introduce notation for describing stress forces. Consider a tiny planar area element dA $\perp$ to x_1 axis at a point P:



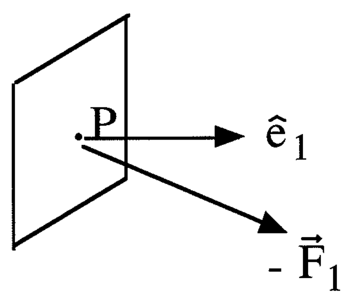


Let $\vec{F}_1$ be stress force exerted across this area by fluid pierced by $\hat{e}_1$ on fluid on other side [So blob of interest is to left of this sfc]



(Can be a "pushing" or "pulling" force. In this picture it's "pushing". In Kundu it's "pulling")

From law of action and reaction (Newton's 3rd law), the stress force exerted across this area by fluid pierced by $-\hat{e}_1$ on fluid on other side is $-\vec{F}_1$.



[Note: can deduce Newton's 3rd law by applying Newton's 2nd law to an infinitesimal fluid blob and considering acceleration to be <u>finite</u>].

$\vec{F}_1$ is not the "1" component of a vector. It's the force on the "x_1 face." Has 3 comps (normal comp, 2 tangential comps).

$$\vec{F}_1 = \left(\boxed{\tau_{11}}\, \hat{e}_1 + \boxed{\tau_{12}}\, \hat{e}_2 + \boxed{\tau_{13}}\, \hat{e}_3 \right) dA = \tau_{1j}\, \hat{e}_j\, dA$$

τ_{11} ↓ normal stress on face 1

τ_{12}, τ_{13} ↓ tangential (shear) stresses on face 1

Stress has units of force per unit area (p. u. a.)
Stress force has units of force (i.e., units of stress times area)