METR 5113, Advanced Atmospheric Dynamics I
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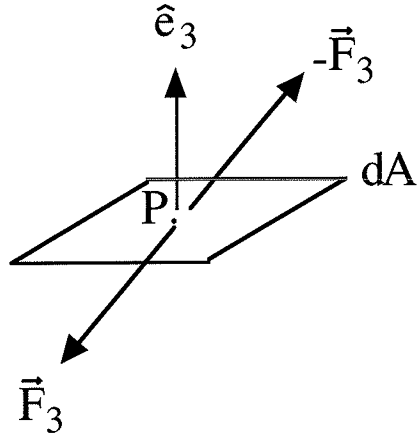
Friday, 21 September 2018 (lecture 14)

- 2 handouts: a test for tensor character, alternate derivation of Cauchy's eqn of motion

Forces (continued)

At same point P consider an area element dA $\perp$ to x_3 axis (dif orientation)



$\vec{F}_3$ is force exerted across dA by fluid pierced by $\hat{e}_3$ on fluid on other side. $-\vec{F}_3$ is force exerted across dA by fluid pierced by $-\hat{e}_3$ on fluid on other side.

$\vec{F}_3$ doesn't have to equal $\vec{F}_1$ even though both are evaluated at same location and time w/ same sized dA (dif orientation of dA)

For force on the i'th face:

$$\vec{F}_i = \left(\tau_{i1}\,\hat{e}_1 + \tau_{i2}\,\hat{e}_2 + \tau_{i3}\,\hat{e}_3\right) dA$$

$$\vec{F}_i = \tau_{ij}\,\hat{e}_j\, dA$$

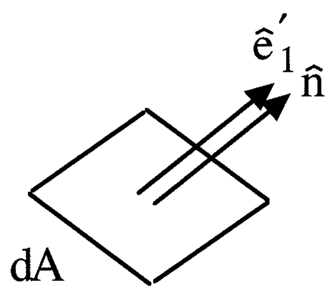
force on i'th face,
not i-comp of a force

τ_{ij} is j-comp of the stress acting on i'th face (face $\perp$ to x_i) following convention above (fluid pierced by $\hat{e}_i$ is do-er).

- Can prove that τ is a 2nd order tensor (stress tensor)

- Can prove that τ is symmetric $\tau_{ij} = \tau_{ji}$.
[for proofs, see Batchelor's "intro to fluid dynamics" along with my handout on a special test for tensor character]

What is the stress force on an area element dA with arbitrary orientation? Let $\hat{n}$ be unit vector $\perp$ to sfc. Introduce new (rotated) coord system s.t. $\hat{e}_1'$ axis is aligned with $\hat{n}$.



So stress force on this $\hat{e}'_1$ - oriented element is:

$\vec{F}_n = \vec{F}_{1'} = \tau'_{1j}\,\hat{e}'_j\,dA$ (comps in new rotated coord system)

Can rewrite it in vector form:

$\vec{F}_n = \tau'_{1j}\,\hat{e}'_j\,dA$ (now use tensor transformation law)

$= C_{k1}\,C_{mj}\,\tau_{km}\,\hat{e}'_j\,dA$ (now use $C_{mj}\hat{e}'_j = \hat{e}_m$)

$= C_{k1}\,\tau_{km}\,\hat{e}_m\,dA$

now use: $C_{k1} = \hat{e}_k \cdot \hat{e}_1' = \hat{e}_k \cdot \hat{n} = n_k$

So $\vec{F}_n = \tau_{km} n_k \hat{e}_m dA$

$= \tau_{mk} n_k \hat{e}_m dA$ (from symmetry of τ)

$= (\tau \cdot \hat{n})_m \hat{e}_m dA$

so $\vec{F}_n = \tau \cdot \hat{n}\, dA$ stress force on area element w/ normal $\hat{n}$.

or: $\boxed{\vec{F} = \tau \cdot \hat{n}\, dA}$ (leave off subscript n with understanding that $\vec{F}$ is still the force on face with normal $\hat{n}$)

<u>Conservation of Momentum</u> [misnomer -- parcel momentum can change]

Newton's 2nd law for a solid particle:

$$\vec{F} = m\vec{a} = m\frac{D\vec{u}}{Dt} = \frac{D(m\vec{u})}{Dt}$$

sum of forces on a particle — rate of change of momentum

So $\dfrac{D(m\vec{u})}{Dt} = \vec{F}$.

Extend it to a fluid blob (not necessarily tiny) w/ mass $\int_{V(t)} \rho dV$

$$\frac{D}{Dt}\int_{V(t)} \rho\vec{u}\, dV = \int_{V(t)} \rho\vec{g}\, dV + \int_{A(t)} \tau \cdot \hat{n}\, dA$$

rate of change of total momentum — body force — stress forces

In components:

$$\frac{D}{Dt}\int_{V(t)} \rho u_i \, dV = \int_{V(t)} \rho g_i \, dV + \int_{A(t)} \tau_{ij} n_j \, dA$$

After applying Gauss th^{m} to the last term:

$$\frac{D}{Dt}\int_{V(t)} \rho u_i \, dV = \int_{V(t)} \left(\rho g_i + \frac{\partial \tau_{ij}}{\partial x_j} \right) dV$$

This is an <u>integral form of momentum conservation</u>.

Recall that for a <u>material volume</u>:

$$\frac{D}{Dt}\int_{V(t)} \rho u_i \, dV = \int_{V(t)} \rho \frac{Du_i}{Dt} \, dV$$

So momentum conservation eqn becomes

$$\therefore \quad \int_{V(t)} \left(\rho \frac{Du_i}{Dt} - \rho g_i - \frac{\partial \tau_{ij}}{\partial x_j} \right) dV = 0$$

Now use <u>DuBois-Reymond lemma</u>: If $\int_{V(t)} (\ ) \, dV = 0$ for <u>arbitrary</u> V(t) then ( ) must be 0 <u>everywhere</u>.

$$\therefore \quad \rho \frac{Du_i}{Dt} - \rho g_i - \frac{\partial \tau_{ij}}{\partial x_j} = 0 , \qquad \text{or,}$$

$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

Cauchy's eq^{n} of motion

- it's a differential form of momentum conservation.

[See handout for a different derivation of Cauchy's eq^{n}]

Cauchy's eq^{n} is underdetermined (more unknowns than eq^{ns}): 3 eq^{ns} in 9 unknowns (3 for u_i, and 6 for τ_{ij}). To solve for u_i, must close the system. Want to relate τ_{ij} to local flow conditions.