METR 5113, Advanced Atmospheric Dynamics I
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$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j} \qquad \text{Cauchy's eq}^n \text{ of motion}$$

Relate τ_{ij} to local flow. Assume a constitutive relation:

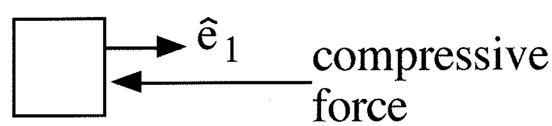
$$\tau_{ij} = -p\,\delta_{ij} + \sigma_{ij} \qquad \text{where} \quad \sigma_{ij} = K_{ijmn}\, e_{mn}\,,$$

p is thermodynamic pressure, σ is deviatoric stress tensor (stress due to motion), K is a 4th order tensor (has $3^4 = 81$ elements), and $-p\,\delta_{ij}$ is an isotropic tensor.

The assumed linear relation btw deviatoric stress and rate of strain, $\sigma_{ij} = K_{ijmn}\, e_{mn}$, is the Newtonian hypothesis.

Need a minus in $-p\,\delta_{ij}$ for normal stresses to be compressive.

Why? Consider case of no motion (so $\sigma_{ij} = 0$). Want surface forces to be inward on parcel. Look at right face of parcel:



Since $\vec{F}_1$ should point in $-\hat{e}_1$ dir^n, we should have $\tau_{11} < 0$ in $\vec{F}_1 = \tau_{11}\hat{e}_1 dA$. Since $p > 0$, need minus sign in front of p in order for $\tau_{11} < 0$: $\tau_{11} = -\boxed{p} + \boxed{\sigma_{11}}$.]

$$\qquad\qquad + \qquad 0$$

Since $\sigma_{ij} = K_{ijmn} e_{mn}$, each comp of σ is related to <u>all</u> comp^s of e

Since fluid is isotropic (random molecular structure) the components of K must be the same in all Cartesian coord systems. So K must be a 4th order isotropic tensor:

$$K_{ijmn} = \lambda\, \delta_{ij}\, \delta_{mn} + \mu\, \delta_{im}\, \delta_{jn} + \gamma\, \delta_{in}\, \delta_{jm}$$

Since τ_{ij} and δ_{ij} are symmetric, σ_{ij} must be symmetric. $\therefore K_{ijmn}$ is symmetric in i, j indices: $K_{ijmn} = K_{jimn}$

$$\therefore \boxed{\lambda\delta_{ij}\delta_{mn}} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm} = \boxed{\lambda\delta_{ji}\delta_{mn}} + \mu\delta_{jm}\delta_{in} + \gamma\delta_{jn}\delta_{im}$$

get cancellation since $\delta_{ij} = \delta_{ji}$

$$\therefore \quad (\mu - \gamma)\, \delta_{im}\, \delta_{jn} + (\gamma - \mu)\, \delta_{in}\, \delta_{jm} = 0$$

$$\therefore \quad \mu = \gamma$$

Using this isotropic K (with $\mu = \gamma$), the stress tensor becomes

$$\tau_{ij} = -p\, \delta_{ij} + [\lambda\, \delta_{ij}\, \delta_{mn} + \mu\, (\delta_{im}\, \delta_{jn} + \delta_{in}\, \delta_{jm})]\, e_{mn}$$

$$= -p\, \delta_{ij} + \lambda\, \delta_{ij}\, e_{mm} + \mu\, (e_{ij} + e_{ji})$$

$e_{ji} \downarrow e_{ij}$ from symmetry

$$= -p\, \delta_{ij} + \lambda\, \delta_{ij} e_{mm} + 2\mu\, e_{ij}$$

Define <u>mechanical pressure</u> p_{mech} to be the (negative) average of the normal stresses (neg ave of diagonal elements of τ):

$$p_{mech} \equiv -\frac{1}{3}\left(\tau_{11} + \tau_{22} + \tau_{33}\right) = -\frac{1}{3}\,\tau_{ii}$$

$$= -\frac{1}{3}(-p\,\delta_{ii} + \lambda\,\delta_{ii}\,\underset{\nabla\cdot\vec{u}}{\underbrace{e_{mm}}} + 2\mu\,\underset{\nabla\cdot\vec{u}}{\underbrace{e_{ii}}})$$

$$= -\frac{1}{3}[-3p + (3\lambda + 2\mu)\,\nabla\cdot\vec{u}]$$

$$\therefore \quad p_{mech} = p - (\lambda + \frac{2}{3}\mu)\,\nabla\cdot\vec{u}$$

Stokes hypothesis: $\lambda + \frac{2}{3}\mu = 0 \rightarrow \lambda = -\frac{2}{3}\mu$

so $p_{mech} = p$, mech pressure = thermo pressure

Holds up well experimentally for air. So, with $\lambda = -\frac{2}{3}\mu$, the constitutive eq^n becomes:

$$\tau_{ij} = -p\,\delta_{ij} - \frac{2}{3}\mu\,\delta_{ij}\nabla\cdot\vec{u} + 2\mu\,e_{ij}$$

Plug this into Cauchy's eq^n of motion:

$$\rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j}(-p\,\delta_{ij} - \frac{2}{3}\mu\,\delta_{ij}\nabla\cdot\vec{u} + 2\mu\,e_{ij})$$

$$\boxed{\rho\frac{Du_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} - \frac{2}{3}\frac{\partial}{\partial x_i}\left(\mu\,\nabla\cdot\vec{u}\right) + 2\frac{\partial}{\partial x_j}\left(\mu\,e_{ij}\right)}$$

Most general form of the Navier-Stokes eq^{ns} of motion.

μ is dynamic viscosity coefficient (a friction coefficient). It's a function of temp:

for air: if T ↑ then μ ↑ ("Friction" in air is really momentum exchange associated w/ random motion of molecules. Motion increases w/ T)

for water: if T ↑ then μ ↓ ("Friction" is due to cohesive forces btw molecules. As T ↑, cohesive forces ↓)

If T gradients are weak, then treat μ as const:

$$\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} - \frac{2}{3} \mu \frac{\partial}{\partial x_i}\left(\nabla \cdot \vec{u}\right) + \boxed{2 \mu \frac{\partial e_{ij}}{\partial x_j}}$$

Note: $$2 \mu \frac{\partial e_{ij}}{\partial x_j} = 2 \mu \frac{\partial}{\partial x_j}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$

$$= \mu \frac{\partial^2 u_i}{\partial x_j^2} + \mu \boxed{\frac{\partial^2 u_j}{\partial x_j \partial x_i}} \text{-------->} \boxed{\frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j}}$$

$$= \mu \nabla^2 u_i + \mu \frac{\partial}{\partial x_i}\left(\nabla \cdot \vec{u}\right)$$

∴ N.S. eq^{ns} become:

$$\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{1}{3} \mu \frac{\partial}{\partial x_i}\left(\nabla \cdot \vec{u}\right) + \mu \nabla^2 u_i$$

In vector form:

$$\rho \frac{D\vec{u}}{Dt} = \rho \vec{g} - \nabla p + \frac{1}{3} \mu \nabla\left(\nabla \cdot \vec{u}\right) + \mu \nabla^2 \vec{u}$$

If flow is <u>incompressible</u> $\left(\nabla \cdot \vec{u} = 0\right)$ then N.S. eq^{ns} become:

$$\rho \frac{D\vec{u}}{Dt} = \rho\vec{g} - \nabla p + \mu \nabla^2 \vec{u} \qquad \div \text{ by } \rho \text{ , get:}$$

$$\boxed{\frac{D\vec{u}}{Dt} = \vec{g} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u}}$$ <u>Navier-Stokes eq^{ns} for incomp flow</u>

$\nu \equiv \frac{\mu}{\rho}$ is kinematic viscosity coeff.

Can rewrite NS eqns for incomp flow as:

$$\frac{\partial\vec{u}}{\partial t} + \left(\vec{u} \cdot \nabla\right)\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g} + \nu\nabla^2\vec{u}$$

local derivative	nonlinear advection	pressure gradient force (pgf)	gravity	diffusion (viscous term)

[time permitting, show how units of μ and ν can be deduced from the NS eqns].