METR 5113, Advanced Atmospheric Dynamics I
Alan Shapiro, Instructor
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1 handout: information about exam 1.

Navier-Stokes eq^{ns} for incomp flow:

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \nabla\right) \vec{u} = -\frac{1}{\rho} \nabla p + \vec{g} + \nu \nabla^2 \vec{u}$$

<u>A simple example: no motion (for all time)</u>

So $u(x, y, z, t) = 0$, $v(x, y, z, t) = 0$, $w(x, y, z, t) = 0$. So all space and time derivs of u, v, w are 0.

$\therefore$ N.S. eq^{ns} become: $0 = -\frac{1}{\rho} \nabla p + \boxed{\vec{g}} \rightarrow -g\hat{k}$

or in components:

z-comp eq^n: $$0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g$$

x-comp eq^n: $$0 = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

y-comp eq^n: $$0 = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$

Integrate x-comp eq^n, get $p = f(y,z)$ [function of integration]
Means p is <u>at most</u> a f^n of y and z.

Integrate y-comp eq^n, get $p = h(x,z)$ [function of integration]
Means p is <u>at most</u> a f^n of x and z.

$\therefore$ f(y, z) = h(x, z)   But this means that there's no y in f(y, z) and no x in h(x,z). So p is a f^n of z <u>only</u>, p = p(z).

Mult z-comp eqn by -ρ then take $\partial/\partial x$ of it:

$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) + g\frac{\partial \rho}{\partial x} = 0$$ now interchange order of differentiation

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) + g\frac{\partial \rho}{\partial x} = 0$$

↓ 0, from x-comp eq^n

$$\therefore \quad \frac{\partial \rho}{\partial x} = 0$$

Similarly can show that

$$\frac{\partial \rho}{\partial y} = 0 .$$

$\therefore$ ρ = ρ(z) only

$\therefore$ z-comp eq^n of motion becomes:

$$\boxed{\frac{dp}{dz} = -\rho(z)\, g}$$ <u>hydrostatic eqn</u>

integrate from arbitrary reference level z_0 to some z of interest,

$$p(z) = p(z_0) - \int_{z_0}^{z} \rho(z')\, g\, dz'$$

If you want, take z_0 to be ∞ (where pressure goes to 0), so

$$p(z) = -\int_{\infty}^{z} \rho(z')\, g\, dz'$$ now swap upper/lower limits and change sign

$$p(z) = \int_{z}^{\infty} \rho(z')\, g\, dz'$$

Interpretation: pressure at height z in a resting atmosphere is the weight of a column of air (of unit horiz area) above z.

______ end of no motion example

The hydrostatic eq^n is often a good approximation to vertical eq^n of motion even when there is motion, especially for large scale or synoptic scale flows. Good approx for shallow motions where $H << L$ [H is characteristic vertical length scale,
L " " horizontal " " .]

[see scale analysis in Pedlosky or section 2.43 of Holton]

For every real atmospheric flow can always associate a "reference" or "base-state" atmosphere which is motionless and hydrostatic w/ density $\bar{\rho}(z)$. Can define $\bar{\rho}(z)$ different ways.

e.g. $\bar{\rho}(z) \equiv$ horizontal ave of density
e.g. $\bar{\rho}(z)$ is a sounding profile

Define base-state pressure $\bar{p}(z)$ through the relations:

$$\frac{\partial \bar{p}}{\partial x} = 0, \quad \frac{\partial \bar{p}}{\partial y} = 0, \quad \frac{\partial \bar{p}}{\partial z} = -\bar{\rho}\, g$$ [could integrate it to get $\bar{p}(z)$]

So $\nabla \bar{p} = -\bar{\rho}\, g\, \hat{k}$

Now define perturbation pressure p' and density ρ':

$$p' \equiv p - \bar{p}(z) \quad \rightarrow \quad p = \bar{p}(z) + p'$$
$$\rho' \equiv \rho - \bar{\rho}(z) \quad \rightarrow \quad \rho = \bar{\rho}(z) + \rho'$$

In practice $\left|\rho'/\rho\right| \approx 0.01 << 1$, $\left|p'/p\right| \approx 0.001 - 0.01 << 1$,

but these small ρ' and p' can be dynamically important.

Rewrite sum of vertical pgf and gravity in vert comp N.S. eq^{ns} in terms of these perturbation quantities:

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = -\frac{1}{\rho}\left(\frac{\partial p}{\partial z} + \rho g\right)$$

$$= -\frac{1}{\rho}\left(\boxed{\frac{\partial \bar{p}}{\partial z}}^{\rightarrow -\bar{\rho} g} + \frac{\partial p'}{\partial z} + \bar{\rho} g + \rho' g\right) \quad \text{now get cancellation}$$

$$\therefore \quad \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = -\frac{1}{\rho}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho} g}$$

vertical comp perturbation pgf; reduced gravity (aka buoyancy)

while horizontal components of pgf become:

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial x}} \quad \text{since } \bar{p} = \bar{p}(z)$$

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{1}{\rho}\frac{\partial p'}{\partial y}} \quad \text{since } \bar{p} = \bar{p}(z)$$

Combine these three into one vector form:

$$-\frac{1}{\rho}\nabla p - g\hat{k} = -\frac{1}{\rho}\nabla p' - \frac{\rho'}{\rho} g\,\hat{k}$$

It's an exact expression (no approximations were made to get it).

Scale analysis of vert eq^n of motion is done to determine which (if any) vertical acceleration terms $\frac{\partial w}{\partial t}, u\frac{\partial w}{\partial x}, v\frac{\partial w}{\partial y}, w\frac{\partial w}{\partial z}$ can be safely neglected. Compare them to perturbation pgf and reduced gravity.

Consider $w\frac{\partial w}{\partial z}$. It's characteristic size is $\frac{W^2}{H}$ where W is the characteristic vertical velocity scale and H is characteristic vertical length scale (H ~ 10 km for deep convective storms and synoptic-scale systems).

In intense t-storm updrafts, W can be on the order of 30 m/s, so

$$w\frac{\partial w}{\partial z} \sim \frac{W^2}{H} = \frac{(30\text{m/s})^2}{10000\text{m}} \approx 10^{-1}\text{ms}^{-2}$$

In run-of-the-mill t-storms, W is on the order of 1-10 m/s, so

$$w\frac{\partial w}{\partial z} \sim \frac{W^2}{H} = \ldots \text{ ranges btw } 10^{-4} - 10^{-2}\text{ms}^{-2}$$

For synoptic-scale flows, W is on order of 1 cm/s = 0.01 m/s, so

$$w\frac{\partial w}{\partial z} \sim \frac{W^2}{H} = \ldots = 10^{-8}\text{ms}^{-2}$$

So how does $w\frac{\partial w}{\partial z}$ compare with pgf and gravity?

$\frac{1}{\rho}\frac{\partial p}{\partial z}$ and g are ~10m/s^2

<u>Much</u> bigger than $w\frac{\partial w}{\partial z}$ for all flows. Can lead to false conclusion that $w\frac{\partial w}{\partial z}$ can always be neglected.

$\frac{1}{\rho}\frac{\partial p'}{\partial z}$ and $\frac{\rho'}{\rho}g$ are ~ 0.1 m/s^2

Bigger than $w\frac{\partial w}{\partial z}$ for synoptic and run-of-mill t-storms but approx same size as $w\frac{\partial w}{\partial z}$ in intense t-storm updrafts. So keep all 3 terms in intense t-storms.

Perturbation pgf/reduced gravity useful for numerical modelling and theoretical work, especially when further approximations are made.

An approximation: if we ignore horizontal variations in density then $\rho = \bar{\rho}$, and $\rho' = 0$. So $-\frac{1}{\rho}\nabla p - g\hat{k} = -\frac{1}{\bar{\rho}}\nabla p'$

Another approximation: if we ignore all density variations then $\rho = \rho_0$, a const, so $\bar{\rho} = \rho_0$, and $\rho' = 0$. So

$$-\frac{1}{\rho}\nabla p - g\hat{k} = -\frac{1}{\rho_0}\nabla p'$$

You also get this equation if the density really is a constant.