METR 5113, Advanced Atmospheric Dynamics I
Alan Shapiro, Instructor
Friday, 28 September 2018 (lecture 17)

Boussinesq Approximation (really 3 different approx)

Real flows are not incompressible, but can be "almost" incomp:

An exact form of mass cons: $\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \vec{u} = 0$. If $\frac{1}{\rho}\frac{D\rho}{Dt}$ is small compared to terms in $\nabla \cdot \vec{u}$, i.e., if $\frac{1}{\rho}\frac{D\rho}{Dt} << \max\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}\right)$, then a good approximation to mass cons^n eq^n is $\nabla \cdot \vec{u} = 0$.

If you make this approximation then you can't return to exact mass cons^n eq^n to deduce $D\rho/Dt = 0$. No double-dipping! Use exact eq^n or approx eq^n but not both (true for any eq^n and its approx). [However, if you approximate 1st Law of Thermo as $D\rho/Dt = 0$, then you can use both $D\rho/Dt = 0$ and $\nabla \cdot \vec{u} = 0$ -- as in our upcoming work w/ gravity waves.]

Boussinesq Approx 1: $\nabla \cdot \vec{u} = 0$ is our mass cons eqn.

Boussinesq Approx 2: material properties μ, c_p held const.

With these two approximations, the N.S. eqns become:

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p - \rho g \hat{k} + \mu \nabla^2 \vec{u}$$

Now define base-state pressure based on constant density ρ_0 (ref atmosphere has const density). So $\rho = \rho_0 + \rho'$. Then $p = \bar{p} + p'$ where $\bar{p}$ is solution of $d\bar{p}/dz = -\rho_0 g$. The NS eqns become:

$$\left(\rho_0 + \rho'\right)\frac{D\vec{u}}{Dt} = -\nabla p' \overbrace{\boxed{-\nabla\bar{p}}\ \boxed{-\rho_0 g\hat{k}}}^{\text{cancellation}} - \rho' g\hat{k} + \mu\nabla^2\vec{u} \qquad \div \text{ by } \rho_0\text{, get:}$$

$$\left(1 + \frac{\rho'}{\rho_0}\right)\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0} g\,\hat{k} + \nu\,\nabla^2\vec{u}$$

ρ' is in inertia term $\left(1 + \frac{\rho'}{\rho_0}\right)\frac{D\vec{u}}{Dt}$ and in gravity term $\frac{\rho'}{\rho_0} g$. If $\frac{\rho'}{\rho_0} << 1$, can safely neglect $\frac{\rho'}{\rho_0}$ compared to 1. In this case can omit $\frac{\rho'}{\rho_0}$ in inertia term but not in gravity term. So we get:

Boussinesq Approx 3: Density is allowed to vary in gravity term but not in inertia term. (i.e., ρ' is neglected in inertia term).

So in the Boussinesq approx, the NS equations reduce to

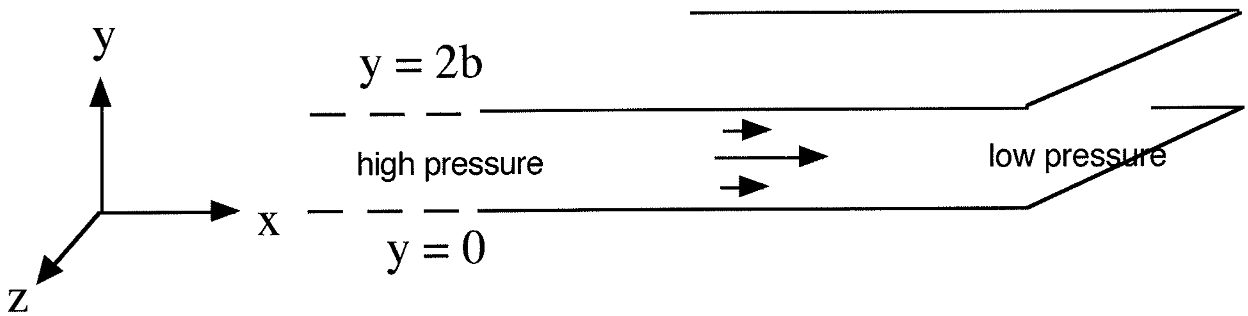
$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0} g\,\hat{k} + \nu\,\nabla^2\vec{u}$$

Exact Solutions of the Incompressible Navier-Stokes eq^{ns}
(in Kundu; Ch 9 of first-fourth ed, Ch8 of 5th ed)

Rare because of non-linearity. Only get sol^{ns} for special cases (usually when nonlinear terms vanish) -- and for laminar flow. The solns are valuable because they show how eqns (our proxy for nature) behave in simple circumstances. Also useful for numerical code validation.

e.g. Planar Poiseuille flow

Unidirectional flow btw 2 infinite parallel plates forced by a pressure gradient force. The plates are stationary. Assume flow is incompressible and ρ is const. Assume flow is in a steady-state. Seek solutions of the form: $u = u(y)$, $v = 0$, $w = 0$.



Need to impose impermeability cond^n at plates: no flow normal to plates. Automatically satisfied in this case:

- at bottom plate $v(0) = 0$ (satisfied since $v = 0$ everywhere)
- at top plate $v(2b) = 0$. (" " " ")

Apply no-slip condition at plates: no flow tangential to plates -- fluid sticks to plates [appropriate at solid bdry in a viscous flow].

No-slip on w is automatically satisfied since $w = 0$ everywhere.
No-slip for u: $u(0) = 0$, $u(2b) = 0$ [sol^n for u must satisfy these]

Incompressibility cond^n:

$$\underbrace{\frac{\partial u}{\partial x}}_{\substack{0 \\ \text{since } u=u(y)}} + \underbrace{\frac{\partial v}{\partial y}}_{\substack{0 \\ v=0}} + \underbrace{\frac{\partial w}{\partial z}}_{\substack{0 \\ w=0}} = 0$$

$\therefore\ 0 = 0$ good, it's satisfied

N.S. eq^{ns} (for incomp flow):

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\hat{k} + \nu\nabla^2\vec{u}$$

or: $$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p' - \frac{\rho'}{\rho}g\hat{k} + \nu\nabla^2\vec{u}$$

but density is assumed to be constant (so $\rho' = 0$) so

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p' + \nu\nabla^2\vec{u}$$

Look at the three components of this eq^n.

x-comp: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

0 (s.s.), 0 ($u=u(y)$), 0 ($v=0$), 0 ($w=0$), ..., 0 ($u=u(y)$), ..., 0 ($u=u(y)$)

y-comp: $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

0 0 0 0 0 0 0

z-comp: $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

0 0 0 0 0 0 0

x-comp: $$0 = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu\frac{d^2 u}{dy^2}$$ (balance btw pgf and friction)

y-comp: $$0 = -\frac{1}{\rho}\frac{\partial p'}{\partial y} \rightarrow p' = f(x,z)$$ at most a f^n of x and z z-comp: $0 = -\frac{1}{\rho}\frac{\partial p'}{\partial z} \quad \rightarrow \quad p' = h(x, y)$ at most a f^n of x and y

Since f(x,z) = h(x,y), must have f(x, z) = f(x), h(x, y) = h(x) (only x dependence). So p'=p'(x), and x-comp NS eq^n becomes

$$0 = -\frac{1}{\rho}\frac{dp'}{dx} + \nu\frac{d^2u}{dy^2}$$

<u>at most</u> a f^n of x — since u = u(y), this is <u>at most</u> a f^n of y

The only way these terms can balance is if they're both <u>constant</u>. So dp'/dx is a const. Call it K ($K \equiv dp'/dx$).

$$\therefore \quad \frac{d^2u}{dy^2} = \frac{K}{\rho\nu}$$

integrate w.r.t. y, get:

$$\frac{du}{dy} = \frac{K}{\rho\nu}y + C,$$

integrate again, get:

(C: const of integration)

$$u = \frac{K}{\rho\nu}\frac{y^2}{2} + Cy + D.$$

(D: another const of integration)

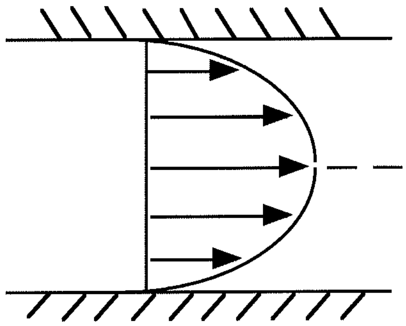
Apply no-slip b.c. for u on lower plate:

$$u(0) = 0 \rightarrow 0 = 0 + 0 + D \quad \therefore \boxed{D = 0}$$

Apply no-slip b.c. for u on upper plate:

$$u(2b) = 0 \rightarrow 0 = \frac{K}{\rho\nu}\frac{(2b)^2}{2} + C(2b) + D \quad \therefore \boxed{C = -\frac{Kb}{\rho\nu}}$$

$$\boxed{u = \frac{K}{\rho\nu}\left(\frac{y^2}{2} - by\right)}$$ a parabolic profile



Verify that u is max at centerline. Define y_{max} by: $\left.\frac{du}{dy}\right|_{y_{max}} = 0$.

$$\therefore \quad \left.\frac{K}{\rho\nu}(y - b)\right|_{y_{max}} = 0 \qquad \therefore \quad y_{max} = b$$

Calculate the maximum value of u:

$$u_{max} = u(y_{max}) = \frac{K}{\rho\nu}\left(\frac{y_{max}{}^2}{2} - by_{max}\right) = \frac{K}{\rho\nu}\left(\frac{b^2}{2} - b^2\right) = -\frac{Kb^2}{2\rho\nu}$$

If $K < 0$ ($dp'/dx < 0$) then $u_{max} > 0$ (flow from high p' toward low p').