METR 5113, Advanced Atmospheric Dynamics I
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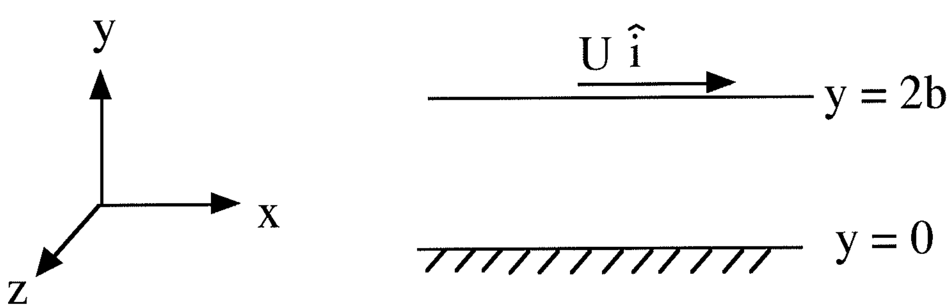
Monday, 1 Rocktober 2018 (lecture 18)

Exact solutions of incompressible Navier-Stokes eq$^{\text{ns}}$ (cont$^{\text{d}}$)

Planar Couette Flow

[Ask students to go through this Couette analysis themselves]

Again consider steady flow btw 2 infinite parallel plates. This time suppose there is no pgf but the top plate is moving with velocity $U\hat{i}$. As before, assume 2-D flow u = u(y), v = 0, w = 0.



Impermeability cond$^{\text{n}}$: v(0) = 0, v(2b) = 0. Automatically satisfied since v = 0 everywhere.

No-slip cond$^{\text{n}}$ on w: w(0) = 0, w(2b) = 0. Automatically satisfied since w = 0 everywhere.

No slip cond$^{\text{n}}$ on u: u(0) = 0 (fluid sticks to stationary plate)
u(2b) = U (fluid sticks to moving plate)
We'll enforce these conditions later.

Can show that incomp cond$^{\text{n}}$ is satisfied everywhere (get 0 = 0). Write down N.S. eq$^{\text{ns}}$ and slaughter terms like before but now all pgf terms are 0 (flow driven by moving plate, not pgf). Get:

x-comp NS eq^n: $\nu \dfrac{d^2u}{dy^2} = 0$

y-comp NS eq^n: $0 = 0$

z-comp NS eq^n: $0 = 0$

int. x-comp eq^n w.r.t. y, get

$\dfrac{du}{dy} = C$, int again w.r.t. y, get

$u = Cy + D$

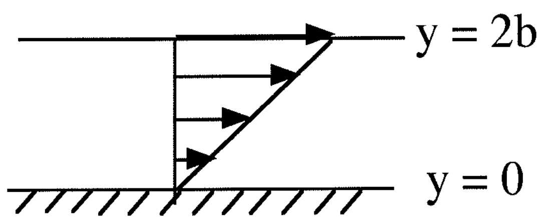
Apply no-slip condition on u on lower plate:

$u(0) = 0 \rightarrow 0 = 0 + D \quad \therefore \boxed{D = 0}$

Apply no-slip condition on u on upper plate:

$u(2b) = U \rightarrow U = C\,2b + \underset{0}{D} \quad \therefore \boxed{C = \dfrac{U}{2b}}$

$\therefore \boxed{u = \dfrac{U}{2b}\,y}$ linear profile



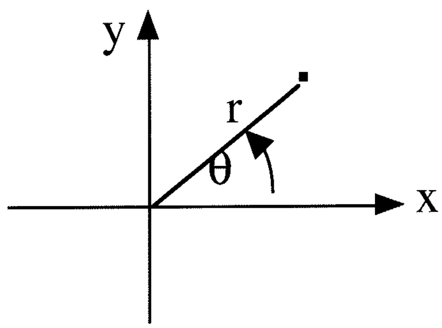
Combination Planar Poiseuille/Couette flow

Get a sol^n for flow due to boundary translation and imposed pgf
[You're invited to ~~a party~~ fill in the details.]

$$u = \frac{U}{2b}y + \frac{K}{\rho\nu}\left(\frac{y^2}{2} - by\right), \quad v = 0, \quad w = 0$$

[for picture, see Kundu, Fig. 9.4 (eds. 1-4), or Fig. 8.4 (5th ed.)]

N.S. eq$^{\underline{ns}}$ in cylindrical coords



$x = r\cos\theta$

$y = r\sin\theta$

$z \perp$ to xy plane

Cartesian	Cylindrical
x, y, z	r, θ, z (r: radial, θ: azimuthal)
u, v, w	u_r, u_θ, u_z or call them u, v, w (not Cartesian!)

r comp NS eq^n:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p'}{\partial r} + \nu\left(\nabla^2 u - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2}\right)$$

θ-comp NS eq^n:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p'}{\partial \theta} + \nu\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right)$$

z-comp NS eq^n:

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + \nu \nabla^2 w$$

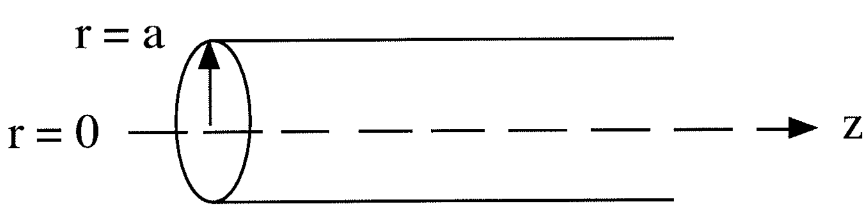
where $$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Incomp cond^n: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,u\right) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

Poiseuille flow

Consider steady unidirectional pressure-driven flow in a pipe of circular cross-section.

Use cylindrical coords. Put z-axis down center of pipe. Let pipe radius be a.



Assume unidirectional flow such that:

$\vec{u} = w\,\hat{k}$, $u = 0$, (no radial flow) $v = 0$ (no azimuthal flow)

Also assume flow is symmetric about pipe axis and indep of downstream direction z. $\therefore$ $w = w(r)$

Impermeability condition: flow component normal to pipe should be 0 (on pipe), i.e., u(a) =0. It's satisfied since u =0 everywhere.

No-slip condition on pipe surface (involves v and w):
v(a) = 0 satisfied since v=0 everywhere
w(a) = 0 ... need to return to this one later.

Incomp cond^n: $\frac{1}{r}\frac{\partial}{\partial r}\left(r\,u\right) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$

0 since u = 0   0 v =0   0 w indep of z

get: 0 + 0 + 0 = 0. okee

N.S. eq^{ns} for this case reduce to:

r comp: $0 = -\frac{1}{\rho}\frac{\partial p'}{\partial r} \quad \rightarrow \quad p' = f(\theta, z)$

θ-comp: $0 = -\frac{1}{\rho r}\frac{\partial p'}{\partial \theta} \quad \rightarrow \quad p' = g(r, z)$

z-comp: $0 = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$

Since f(θ, z) = g(r, z) must have f(θ, z) = f(z) and g(r,z) = g(z) -- only z dependence is possible. So p' = p'(z).

Rewrite z-comp eq^n with ordinary derivatives,

$$0 = -\frac{1}{\rho}\frac{dp'}{dz} + \frac{\nu}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)$$

at most a f^n of z   at most a f^n of r

But a f^n of z can't equal a f^n of r unless the functions are <u>const.</u>

Mult by r/ν:

$$\frac{d}{dr}\left(r\,\frac{dw}{dr}\right) = \frac{r}{\rho\nu}\frac{dp'}{dz} = \frac{r}{\mu}\frac{dp'}{dz} \qquad \text{int w.r.t. r,}$$

$$r\,\frac{dw}{dr} = \frac{r^2}{2\mu}\frac{dp'}{dz} + A \qquad \div \text{ by r ,}$$

$$\frac{dw}{dr} = \frac{r}{2\mu}\frac{dp'}{dz} + \frac{A}{r} \qquad \text{int w.r.t r,}$$

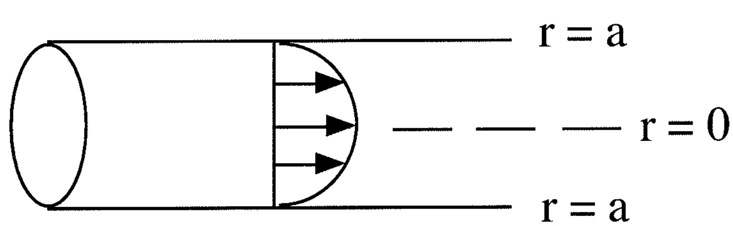
$$w(r) = \frac{r^2}{4\mu}\frac{dp'}{dz} + A\ln r + B$$

Want w to be finite along centerline r=0 (a finiteness "boundary condition") ∴ take $\boxed{A = 0}$ to avoid singularity.

No-slip b.c. on pipe wall: $w(a) = 0$.

$$\therefore \quad 0 = \frac{a^2}{4\mu}\frac{dp'}{dz} + \underset{\downarrow \atop 0}{A}\ln a + B, \qquad \therefore \quad \boxed{B = -\frac{a^2}{4\mu}\frac{dp'}{dz}}$$

$$\therefore \quad \boxed{w(r) = -\frac{1}{4\mu}\frac{dp'}{dz}\left(a^2 - r^2\right)} \qquad \text{a parabolic profile}$$



Volume flux through pipe:

$$Q \equiv \int \vec{u}\cdot\hat{n}dA = \int_0^{2\pi}\int_0^a w\, r dr\, d\theta = 2\pi \int_0^a w\, r\, dr$$

$$= -\frac{2\pi}{4\mu}\frac{dp'}{dz}\boxed{\int_0^a (a^2 - r^2)\, r\, dr} \rightarrow a^4/4$$

$$\therefore \boxed{Q = -\frac{\pi}{8\mu}\frac{dp'}{dz}a^4} \quad \text{Hagen-Poiseuille law}$$

- obtained theoretically by Stokes but he didn't publish it because it didn't agree w/ his experiments. His experiments were <u>turbulent</u> but the theory is for <u>laminar flow.</u> [see pg 92 of Truesdell's "Six lectures on modern natural philosophy"]

- HP law discovered experimentally by Hagen and Poiseuille. [1993 Ann. Rev. Fluid Mech. article on Poiseuille's experiments]

- Experimental confirmation of HP law confirms appropriateness of NS eq^{ns} as governing eq^{ns} of motion (at least for liquid). Confirms F = ma, Newtonian hypothesis and no slip condition.

- volume flux is tremendously sensitive to pipe radius ($Q \propto a^4$)

- $\frac{dp'}{dz} \propto \frac{Q}{a^4}$. In case of <u>blood flow,</u> body tries to maintain a fixed Q. If a ↓ (e.g. due to cholesterol deposits) then $\frac{dp'}{dz}$ goes <u>way up</u> to maintain Q. This is why cholesterol deposits are often associated w/ high blood pressure.

HP law breaks down when flow becomes turbulent (as in Stokes case). Then N.S. eq^n are <u>still valid</u> but our starting assumptions are violated (flow is unsteady, not unidirectional, no symmetry).