METR 5113, Advanced Atmospheric Dynamics I
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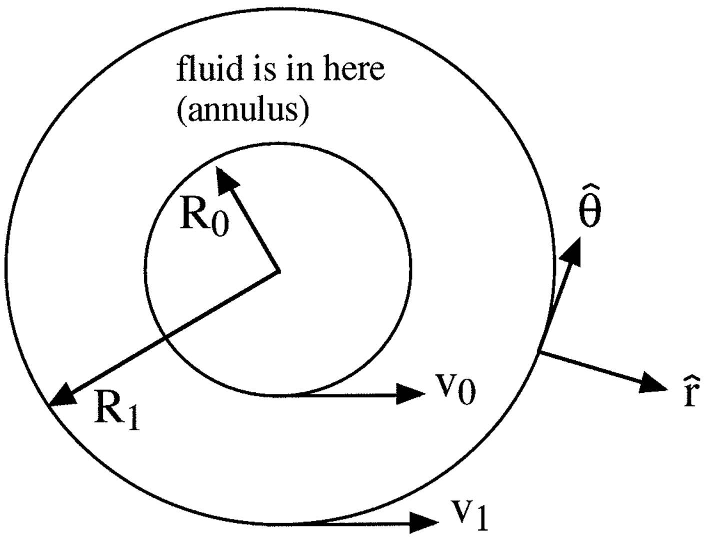
Wednesday, 3 Rocktober 2018 (lecture 19)

- 2 handouts: p.s. 3, unsteady planar Couette + Poiseuille flows

Couette flow

Consider steady, axisymmetric, incompressible flow btw 2 concentric rotating circular cylinders. Assume ρ is const.



$v_0 = R_0 \Omega_0$ where $\Omega_0 \equiv \frac{D\theta}{Dt}(R_0)$ is ang velocity at $r=R_0$

$v_1 = R_1 \Omega_1$ where $\Omega_1 \equiv \frac{D\theta}{Dt}(R_1)$ " " " " $r=R_1$

Assume u = w = 0 (no radial or vertical flow)
v = v(r) (axisymmetry -- no θ dependence)

Incomp cond^n: $\frac{1}{r}\frac{\partial}{\partial r}(r\,u) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$

0	0	0
since u = 0	v = v(r)	w=0

so 0 + 0 + 0 = 0. okee, it works

N.S. eq^{ns} reduce to:

r comp: $\boxed{-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p'}{\partial r}}$ <--- **cyclostrophic balance**

θ-comp: $0 = -\frac{1}{\rho r}\frac{\partial p'}{\partial \theta} + \nu\left(\nabla^2 v - \frac{v}{r^2}\right)$

z-comp: $0 = -\frac{1}{\rho}\frac{\partial p'}{\partial z}$

Since $v = v(r)$, the θ-comp eq^n yields:

$$\frac{\partial p'}{\partial \theta} = f(r)$$ integrate. w.r.t. θ, get

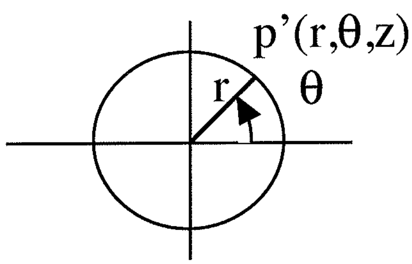
$$p' = \theta f(r) + b(r,z)$$ b is function of integration

But z-comp eq^n says $\frac{\partial p'}{\partial z} = 0$

$$\therefore \frac{\partial b}{\partial z} = 0 \quad \therefore b(r,z) = b(r) \text{ only}$$

$$\therefore p' = \theta f(r) + b(r)$$

Want p' to be <u>periodic</u> in θ. [Go around by 2π and end up at same point -- should get same p'.]



$$p'(r, \theta, z) = p'(r, \theta + 2\pi, z)$$

$\therefore\ \boxed{\theta\, f(r)} + \boxed{b(r)} = (\boxed{\theta} + 2\pi)\, f(r) + \boxed{b(r)}$ cancellation

$\therefore\ 2\pi\, f(r) = 0 \qquad \therefore\ f(r) = 0$

$\therefore\ p' = b(r)$ no θ dependence -- p' is axisymmetric

Now go to radial eq^n (and note that $\frac{\partial p'}{\partial r} = \frac{dp'}{dr}$):

$$-\frac{v^2}{r} = -\frac{1}{\rho}\frac{dp'}{dr}$$

int w.r.t. r:

$$p'(r) = p'_0 + \int_{R_0}^{r} \rho \frac{v^2(R)}{R} dR\,, \qquad p'_0 \text{ is const of int} = p'(R_0)$$

Once we know v, can get p' from this formula. Since integrand is positive, p' $\uparrow$ as r $\uparrow$. Perturbation pressure increases outward.

Get v from azimuthal eq^n:

$$0 = \nabla^2 v - \frac{v}{r^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}}_{\downarrow\ 0} + \underbrace{\frac{\partial^2 v}{\partial z^2}}_{\downarrow\ 0} - \frac{v}{r^2}$$

since $v = v(r)$

$$\therefore\ \frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = 0$$

2nd order linear ode w/ variable coeff in which each denom "looks like" r^2 -- an Euler eq^n. Seek sol^{ns} of the form: $v = r^n$

$$\therefore \frac{dv}{dr} = n\, r^{n-1} \qquad \therefore \frac{d^2v}{dr^2} = n\,(n-1)\, r^{n-2}$$

plug these into the ode, get:

$$n(n-1)\, r^{n-2} + n\, r^{n-2} - r^{n-2} = 0$$

$$\therefore \; n(n-1) + n - 1 = 0$$

$$\therefore \; n^2 = 1$$

$$\therefore \; n = 1 \text{ or } n = -1$$

So the general sol^n of the ode is:

$$v = \underset{\text{solid body vortex}}{A\, r} + \underset{\text{vr vortex}}{\frac{B}{r}}$$

i.e., solution at every point is a combination (sum) of solid body vortex and vr vortex (dif than Rankine vortex which is <u>all</u> solid body vortex in inner region and <u>all</u> vr vortex in outer region).

Apply no-slip b.c. to get A, B:

On inner cylinder: $v(R_0) = \Omega_0 R_0$

$$\therefore \quad \boxed{\Omega_0 R_0 = A\, R_0 + \frac{B}{R_0}}$$

On outer cylinder: $v(R_1) = \Omega_1 R_1$

$$\therefore \quad \boxed{\Omega_1 R_1 = A R_1 + \frac{B}{R_1}}$$

These are 2 linear algebraic eq^{ns} for A, B. Solve them to get:

$$A = \frac{\Omega_1 R_1^2 - \Omega_0 R_0^2}{R_1^2 - R_0^2}, \qquad B = \frac{R_1^2 R_0^2 \left(\Omega_0 - \Omega_1\right)}{R_1^2 - R_0^2}$$

<u>Decay of a line vortex</u> (an unsteady problem)

[solution involves "dimensional analysis" -- an extremely powerful technique in boundary layer theory and turbulence.]

Recall the vr vortex: $v = \dfrac{\Gamma}{2\pi r}$ $\quad (u = 0, \quad w = 0)$

This is actually a sol^n of the N.S. eq^{ns} if no boundaries are present (provided p' satisfies radial NS eq^n). This flow can be generated in a lab by twirling a wire very quickly.
Lets see how this vortex decays in time (wire stops twirling at t=0) -- this unsteady problem may be relevant for the temporal decay of natural small scale vortices (e.g. dust devils, turbulent eddies) and wing-tip vortices behind aircraft.

Assume u=0, w=0, ρ=const and axisymmetry. Many terms drop from N.S. eq^{ns}. Radial comp eqn reduces to cyclostrophic balance (with v being time dependent). Integrate it w.r.t. r from far away ($r = \infty$), where p' vanishes, to arbitrary r, get:

$$p'(r,t) = -\int_r^{\infty} \rho \frac{v^2(R,t)}{R} \, dR$$

Azimuthal N.S. eq^n reduces to:

$$(*) \quad \frac{\partial v}{\partial t} = \nu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} \right)$$

Initial $cond^n$ (I.C.): $v(r, 0) = \dfrac{\Gamma}{2\pi r}$

Boundary $cond^{ns}$ (B.C.):

- At $r = 0$ want $v(0, t) = 0$ for $t > 0$ (finite stress at $r = 0$).
- Far away, the flow doesn't know the vortex has been "turned off": $\lim_{r\to\infty} v(r, t) = \lim_{r\to\infty} \dfrac{\Gamma}{2\pi r}$ for $t > 0$