METR 5113, Advanced Atmospheric Dynamics I
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1 handout: my explanation of Fig. 7.3 of Kundu (3rd ed)

Einstein summation convention: If an index appears twice in the same term, a summation over that index is implied and the summation sign can be omitted. For example:

$$x'_j = \sum_{k=1}^{3} C_{kj} x_k \quad \text{becomes:} \quad x'_j = C_{kj} x_k$$

An index that appears twice in same term is a repeated index, also known as a dummy index. (in eqn above k is a repeated/dummy index).

An index that appears only once per term is a free index (in eqn above j is a free index).

Repeated indices can be renamed. Use any symbol you want as long as it doesn't conflict with other indices. e.g. you can write $x'_j = C_{kj} x_k$ as $x'_j = C_{mj} x_m$ but NOT as $x'_j = C_{jj} x_j$
Can't have 3 or more of same index in same term!

A free index in one term must appear in all terms. Free indices can be renamed, but if you rename a free index in one term you must rename it in all terms, e.g., $x'_l = C_{kl} x_k$ means $x'_m = C_{km} x_k$

Question: How are components of $\vec{x}$ in old system related to comps of $\vec{x}$ in new system?

$$x_i = \vec{x} \cdot \hat{e}_i = x'_j \hat{e}'_j \cdot \hat{e}_i = C_{ij} x'_j$$

So $x_i = C_{ij} x'_j$

Can rewrite it as,

$x_i = C_{ik} x'_k$ (renamed dummy indices)

or: $x_j = C_{jk} x'_k$ (renamed free index -- in every term)

Compare with previous result: $x'_j = C_{kj} x_k$

Now consider Newton's 2nd law: $\vec{F} = m\vec{a}$, or $\vec{F} = m \dfrac{d^2\vec{x}}{dt^2}$.

It has rotational invariance, meaning that the generic form of the components of this eqn does not change under a coord rotation.

In original system:

$$(1) \quad F_i = m \frac{d^2 x_i}{dt^2}$$ (i = 1, 2, 3 in turn)

In rotated system:

$$(2) \quad F'_j = m \frac{d^2 x'_j}{dt^2}$$ (j = 1, 2, 3 in turn. Can use i instead of j here.)

Now we'll show how rotational invariance yields a coordinate transformation law for the comps of $\vec{F}$. Starting with (2):

$$F'_j = m\frac{d^2 \boxed{x'_j}}{dt^2} = m\frac{d^2 C_{ij}x_i}{dt^2} = C_{ij}\, m\frac{d^2 x_i}{dt^2} = C_{ij}F_i$$

So: $F'_j = C_{ij}F_i$

It's analogous to $x'_j = C_{kj}x_k$ [or $x'_j = C_{ij}x_i$] . So, comps of $\vec{F}$ transform <u>with the same rule</u> as the comps of pos^n vector $\vec{x}$.

[Alt derivation: mult (1) by C_{ij} (so, a sum over i). Equivalent to multiplying <u>each</u> of 3 eq^{ns} by a cosine, then summing the 3 eq^{ns} :

$$C_{ij}F_i = m\frac{d^2 C_{ij}x_i}{dt^2}, \quad \text{now use } x'_j = C_{ij}x_i \text{, get:}$$

$$C_{ij}F_i = m\frac{d^2 x'_j}{dt^2} .$$

Compare this result with (2), get $F'_j = C_{ij}F_i$, as before.]

<u>General definition of a vector</u>: a quantity $\vec{P}$ is a <u>vector</u> if its comps transform like the pos^n vector under a rotation of coords:

$$\boxed{P'_j = C_{ij}P_i} \qquad \text{(3 eqns; } j = 1, 2, 3 \text{ in turn)}$$

Vectors are 1st order tensors.

Scalars are zeroth order tensors.

Example: Prove that fluid velocity $\vec{u} \equiv d\vec{x}/dt$ is really a vector.

In original system, the components of $\vec{u}$ are: $u_i = \frac{dx_i}{dt}$

In the rotated system, the components of $\vec{u}$ are: $u_j' = \frac{dx_j'}{dt}$

so $u_j' = \frac{dx_j'}{dt} = \frac{dC_{ij}x_i}{dt} = C_{ij}\frac{dx_i}{dt} = C_{ij}u_i$

so $u_j' = C_{ij}u_i$ $\quad\therefore\ \vec{u}$ is a vector.

Example: Let $\vec{P} = \begin{pmatrix} 2.50 \\ 3.95 \\ 3.25 \end{pmatrix}$ be price of beef jerky at 3 gas stations.

The comps don't change under a coord rotation, i.e., $P_j' = P_j$.
Can prove $\vec{P}$ isn't a vector using proof by contradiction: assume comps of $\vec{P}$ transform according to the transformation rule for first order tensors, $P_j' = C_{ij}P_i$, then find a coord rotation for which this rule is violated -- meaning the rule isn't a rule and $\vec{P}$ isn't a vector. [Many ways to get a contradiction. For example, with the first two coords rotated clockwise by 90° and the third unchanged, get $C_{12} = 1$, $C_{21} = -1$, and all other $C_{ij} = 0$. For j=1 the "rule" yields $P_1' = C_{i1}P_i = C_{11}P_1 + C_{21}P_2 + C_{31}P_3 = -P_2$, which can't be true because it violates the fact that $P_1' = P_1$.]

Example: "phase velocity" is not a vector --see handout.

A quantity Q is a <u>2nd order tensor</u> if its components transform under a coordinate rotation as:

dummy indices

$$Q'_{mn} = C_{im}\,C_{jn}\,Q_{ij}$$

free indices

2 free indices, m and n.
2 dummy (repeated) indices, i and j.

This defining eq^n for Q is really 9 eq^{ns} because m can be 1, 2 or 3
and n " " 1, 2 or 3

So, considering all 9 possibilities, get 9 equations.

Look at one of those eqns, $Q'_{23} = C_{i2}C_{j3}Q_{ij}$. Expand it out:

$$\begin{aligned} Q'_{23} = {} & C_{12}C_{13}Q_{11} + C_{12}C_{23}Q_{12} + C_{12}C_{33}Q_{13} \\ & + C_{22}C_{13}Q_{21} + C_{22}C_{23}Q_{22} + C_{22}C_{33}Q_{23} \\ & + C_{32}C_{13}Q_{31} + C_{32}C_{23}Q_{32} + C_{32}C_{33}Q_{33} \end{aligned}$$

<u>Example:</u> Suppose a quantity Q has components $Q_{ij} = u_i u_j$ where u is a vector. Prove that Q is a 2nd order tensor.

$$Q'_{mn} = u'_m u'_n = C_{im}u_i C_{jn}u_j = C_{im}C_{jn}u_i u_j = C_{im}C_{jn}Q_{ij}$$

done!

A quantity P is a 4th order tensor if its components transform under a coord rotation as:

$$P'_{mnpq} = C_{im} C_{jn} C_{kp} C_{lq} P_{ijkl}$$

81 eq^{ns} w/ 81 terms on rhs of each eq^{n}.