METR 5113, Advanced Atmospheric Dynamics I
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- **1 Handout: decaying line vortex (Kundu figure)**

Decaying line vortex (cont^d)

$$(*) \quad \frac{\partial v}{\partial t} = \nu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} \right)$$

I.C.: $v(r,0) = \dfrac{\Gamma}{2\pi r}$

B.C.: - at r = 0: $v(0,t) = 0$ (for $t > 0$)

- far away: $\lim\limits_{r\to\infty} v(r,t) = \dfrac{\Gamma}{2\pi r}$ (for $t > 0$)

We'll use dimensional analysis to help solve the problem. Sol^n must be of the form: $v = f(r, t, \nu, \Gamma)$ since no other parameters enter the problem (i.e., appear in (*) or B.C. or I.C.)

Rewrite the problem in terms of $V \equiv \dfrac{v}{\Gamma}$.

÷ (*) and I.C. and B.C. by Γ, get:

$$(**) \qquad \frac{\partial V}{\partial t} = \nu \left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2} \right)$$

I.C.: $V(r,0) = \frac{1}{2\pi r}$

B.C.: $V(0,t) = 0$ $(t > 0)$

$$\lim_{r\to\infty} V(r,t) = \lim_{r\to\infty} \frac{1}{2\pi r} \qquad (t > 0)$$

Note that Γ does not appear in (**) or I.C. or B.C. <u>It has been removed from the problem</u>.

$\therefore\ V = g(r, t, \nu)$

What are dimensions of V? $[V] = \frac{[\nu]}{[\Gamma]} = \frac{\frac{L}{T}}{\frac{L}{T}L} = \frac{1}{L}$

So rV is <u>dimensionless</u>. Since V is at most a f^n of r, t, ν, so is rV. So $rV = h(r, t, \nu)$ where h is a <u>dimensionless</u> combo of r, t, ν. Since $[r] = L$, $[t] = T$, $[\nu] = L^2/T$, some dimensionless combos are: $\frac{\nu t}{r^2}$, $\frac{r^2}{\nu t}$, $\frac{r}{\sqrt{\nu t}}$, $\sin\left(\frac{r^2}{\nu t}\right)$, etc .

Let's work with $\frac{r^2}{4\nu t}$ [Any choice would work. The "4" is convenient but not essential.]

$\therefore\ rV = h\left(\frac{r^2}{4\nu t}\right)$

Introduce new dependent variable: $\lambda \equiv rV$

and new independent variable: $\eta \equiv \frac{r^2}{4\nu t}$

$\therefore\ \lambda = h(\eta)$

Want to convert (**) into an eq^n for λ in terms of η.

For upcoming use:

$$\frac{\partial\eta}{\partial t} = -\frac{r^2}{4\nu t^2}, \qquad \frac{\partial\eta}{\partial r} = \frac{2r}{4\nu t} = \frac{r}{2\nu t}$$

Look at terms in (**):

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\frac{\lambda}{r} = \frac{1}{r}\frac{\partial\lambda}{\partial t} = \frac{1}{r}\frac{d\lambda}{d\eta}\boxed{\frac{\partial\eta}{\partial t}} = -\frac{d\lambda}{d\eta}\frac{r}{4\nu t^2}$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\frac{\lambda}{r} = \frac{1}{r}\frac{\partial\lambda}{\partial r} + \lambda\frac{\partial}{\partial r}\frac{1}{r} = \frac{1}{r}\frac{d\lambda}{d\eta}\boxed{\frac{\partial\eta}{\partial r}} - \frac{\lambda}{r^2}$$

$$= \frac{1}{2\nu t}\frac{d\lambda}{d\eta} - \frac{\lambda}{r^2}$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{\partial}{\partial r}\boxed{\frac{\partial V}{\partial r}} = \frac{1}{2\nu t}\frac{\partial}{\partial r}\frac{d\lambda}{d\eta} - \frac{1}{r^2}\frac{\partial\lambda}{\partial r} - \lambda\frac{\partial}{\partial r}\frac{1}{r^2}$$

sub in from above

$$= \frac{1}{2\nu t}\frac{d^2\lambda}{d\eta^2}\frac{\partial\eta}{\partial r} - \frac{1}{r^2}\frac{d\lambda}{d\eta}\frac{\partial\eta}{\partial r} + \frac{2\lambda}{r^3}$$

$$= \frac{r}{4\nu^2 t^2}\frac{d^2\lambda}{d\eta^2} - \frac{1}{2r\nu t}\frac{d\lambda}{d\eta} + \frac{2\lambda}{r^3}$$

Sub all terms into (**):

$$-\frac{d\lambda}{d\eta}\frac{r}{4\nu t^2} = \nu\left(\frac{r}{4\nu^2 t^2}\frac{d^2\lambda}{d\eta^2}\boxed{-\frac{1}{2r\nu t}\frac{d\lambda}{d\eta}+\frac{2\lambda}{r^3}+\frac{1}{2r\nu t}\frac{d\lambda}{d\eta}-\frac{\lambda}{r^3}-\frac{\lambda}{r^3}}\right)$$

all these terms cancel - yay!

$$\therefore \quad -\frac{d\lambda}{d\eta}\frac{r}{4\nu t^2} = \frac{r}{4\nu t^2}\frac{d^2\lambda}{d\eta^2}$$

$$\therefore \quad \boxed{\frac{d^2\lambda}{d\eta^2}+\frac{d\lambda}{d\eta} = 0}$$ 2nd order linear const coeff <u>ode</u> (not pde)

Need to rewrite I.C. and B.C. into conditions on $\lambda(\eta)$.

I.C.: $\quad V(r,0) = \frac{1}{2\pi r}$.

$$\therefore \text{ at } t=0,\ \lambda = r\,V(r,0) = \frac{1}{2\pi}$$

But $\lim_{t\to 0}\eta = \lim_{t\to 0}\frac{r^2}{4\nu t} = \infty$. So $\boxed{\lambda(\infty) = \frac{1}{2\pi}}$

Now rewrite far-away b.c.:

$$\lim_{r\to\infty} V(r,t) = \lim_{r\to\infty}\frac{1}{2\pi r}$$

$$\therefore \quad \lim_{r\to\infty}\lambda = \frac{1}{2\pi}$$

But as $r\to\infty$, get $\eta\to\infty$ $\quad\therefore$ $\boxed{\lambda(\infty) = \frac{1}{2\pi}}$

It's the same as the I.C. (above). So in terms of η, the <u>initial condition</u> and far-away <u>b.c.</u> are the <u>same condition</u>!

B.C. at r=0 (η=0): $V(0,t) = 0$ $\therefore$ $\boxed{\lambda(0) = 0}$.

So want to solve $\dfrac{d^2\lambda}{d\eta^2} + \dfrac{d\lambda}{d\eta} = 0$ with $\lambda(0) = 0$, $\lambda(\infty) = \dfrac{1}{2\pi}$

Integrate the ode:

$$\frac{d\lambda}{d\eta} + \lambda = A \quad \text{first order linear const coeff ode}$$

Can use separation of variables or use integrating factor (I'll use int factor, go ahead and try sep variables). Mult by e^{η}

$$\frac{d(\lambda e^{\eta})}{d\eta} = Ae^{\eta} \qquad \text{integrate it}$$

$$\lambda e^{\eta} = Ae^{\eta} + B \qquad \text{mult by } e^{-\eta}$$

$$\therefore \quad \lambda = A + B\, e^{-\eta}$$

Now impose B.C./I.C. to get A and B

$\lambda(0) = 0 \rightarrow \quad 0 = A + B \qquad$ So $B = -A$

$$\lambda(\infty) = \frac{1}{2\pi} \rightarrow \frac{1}{2\pi} = A + B \cdot 0$$

$$\therefore \quad A = \frac{1}{2\pi}, \quad B = -\frac{1}{2\pi}$$

$$\therefore \quad \lambda = \frac{1}{2\pi}(1 - e^{-\eta})$$

$$\therefore \quad \boxed{r\,V} = \frac{1}{2\pi}\left[1 - \exp(-\frac{r^2}{4\nu t})\right] \qquad \text{now mult by } \Gamma/r$$

$\downarrow$

$r\,v/\Gamma$

$$\boxed{v = \frac{\Gamma}{2\pi r}\left[1 - \exp(-\frac{r^2}{4\nu t})\right]}$$

$\boxed{u = 0,\ w = 0,\ p' \text{ from integral of centripetal acceleration}}$

- examine graph of solution (handout) from Kundu. Looks like a solid body vortex in close and a vr-vortex far away. Confirm it:

For small r make Taylor approx,

$$\exp(-\frac{r^2}{4\nu t}) \approx 1 - \frac{r^2}{4\nu t}$$

$$\therefore \quad v \approx \frac{\Gamma}{2\pi r}\left[1 - (1 - \frac{r^2}{4\nu t})\right] = \frac{\Gamma}{2\pi r}\,\frac{r^2}{4\nu t}$$

$$\therefore \quad v \approx \frac{\Gamma}{8\pi\nu t}\,r$$

So, for small r, solution is approximately a solid body vortex with ang velocity $\frac{\Gamma}{8\pi\nu t}$. Note that ang velocity $\downarrow$ as t $\uparrow$. Also, larger viscosities are associated w/ smaller ang velocity.

For large r, $\exp(-\frac{r^2}{4\nu t}) << 1$

$\therefore \quad v \approx \frac{\Gamma}{2\pi r}$ so it behaves like a vr vortex far away.