METR 5113, Advanced Atmospheric Dynamics I
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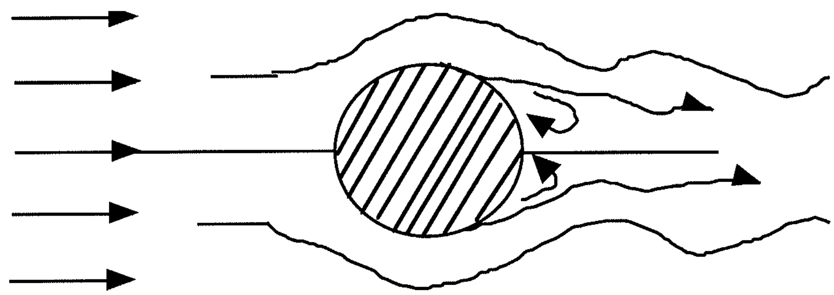
- **2 handouts:** 2 pix from Van Dyke's "Album of fluid motion"

In some cases friction is negligible, at least for part of domain. e.g. flow around a telephone pole:



Friction important in a thin boundary layer (b.l.) adjacent to solid boundary. Can get boundary layer separation downstream (in the lee) of solid object: slow flow near bdry erupts into main flow and leads to a wake of eddies (vortices) and turbulence in the lee. Friction is important in b.l. (next to object) and in wake. <u>But outside of b.l./wake region, friction is not important.</u>

- see handouts on flows around cylinders (from Van Dyke).

<u>Frictionless flow (Inviscid flow)</u>

In areas where friction is not important, can throw out the friction term in the Navier-Stokes equations, get:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\,\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g}} \qquad \underline{\text{Euler's eq}^{\text{ns}}\text{ of motion}}$$

This is eq^{n} of motion for <u>inviscid</u> flows (frictionless flows). Valid for compressible or incompressible flows.

Highest order spatial derivative in N.S. eq^{ns} is 2 (viscous term involves Laplacian, a 2nd order operator). In Euler eq^{ns}, highest order spatial derivative is 1 (advection terms). In general, the higher the order of the d.e., the more b.c.s needed to solve it. At a solid bdry, appropriate b.c. for N.S. eq^{ns} are impermeability and no-slip, but appropriate b.c. for Euler eq^{ns} is just impermeability.

Various forms of Euler Eq^{ns}

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \nabla\right) \vec{u} = -\frac{1}{\rho}\nabla p + \vec{g} \quad \text{(here p is full pressure)}$$

Use vector identity: $\left(\vec{u} \cdot \nabla\right)\vec{u} = \nabla\frac{q^2}{2} + \vec{\omega} \times \vec{u}$, where $q^2 \equiv \vec{u} \cdot \vec{u}$ ($q = \sqrt{\vec{u} \cdot \vec{u}}$ is wind speed).

Also use: $\vec{g} = -g\,\hat{k} = -\nabla gz \qquad (g \equiv |\vec{g}|)$

$$\therefore \quad \frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{q^2}{2} + gz\right) + \frac{1}{\rho}\nabla p + \vec{\omega} \times \vec{u} = 0$$

If the flow is barotropic ($\rho = \rho(p)$) then

$$\frac{1}{\rho}\nabla p = \nabla \int \frac{dp}{\rho} \quad \text{where} \int \frac{dp}{\rho} = \int_{p(\vec{x}_0)}^{p(\vec{x})} \frac{dp}{\rho} \text{ is a line integral}$$

Proof: For barotropic flow, $\frac{1}{\rho}\nabla p$ is $\frac{1}{\rho(p)}\nabla p$. Want to write it as ∇(f^n of p only). Want to find a $G = G(p)$ such that $\boxed{\nabla G = \frac{1}{\rho}\nabla p}$.

From chain rule: $\boxed{\nabla G = \frac{dG}{dp}\nabla p}$. For both to be true we must have $\frac{dG}{dp} = \frac{1}{\rho}$. Integrate w.r.t. p along any line from an arbitrary start point $\vec{x}_0$ to an end point $\vec{x}$: $G(p) = \int_{p(\vec{x}_0)}^{p(\vec{x})} \frac{dp}{\rho}$.

So: $$\boxed{\frac{1}{\rho}\nabla p = \nabla \int_{p(\vec{x}_0)}^{p(\vec{x})} \frac{dp}{\rho}}$$

So with barotropic assumption, Euler's eq^{ns} become:

$$\therefore \quad \frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{q^2}{2} + \int \frac{dp}{\rho} + gz\right) + \vec{\omega} \times \vec{u} = 0$$

<u>Can "integrate" Euler eq^{ns} in 2 Special Cases:</u>

Case 1: If flow is <u>steady and barotropic</u> (and inviscid) then:

$$\therefore \quad \nabla\left(\frac{q^2}{2} + \int \frac{dp}{\rho} + gz\right) + \vec{\omega} \times \vec{u} = 0$$

Let $\vec{dr}$ be an element of streamline. Take $\vec{dr} \cdot$ Euler eq^n, get:

$$\vec{dr} \cdot \nabla\left(\frac{q^2}{2} + \int \frac{dp}{\rho} + gz\right) + \boxed{\vec{dr} \cdot (\vec{\omega} \times \vec{u})} = 0$$

$\downarrow$

0 [$\vec{\omega} \times \vec{u}$ is $\perp$ $\vec{u}$, and $\vec{dr}$ is parallel to $\vec{u}$ so $\vec{\omega} \times \vec{u}$ is $\perp$ $\vec{dr}$]

$$\therefore \quad \vec{dr} \cdot \nabla\left(\frac{q^2}{2} + \int \frac{dp}{\rho} + gz\right) = 0$$

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The tiny change (differential) of Q over a tiny distance $\vec{dl} = dx_i \hat{e}_i$ is: $dQ = \frac{\partial Q}{\partial x_i} dx_i = \nabla Q \cdot \vec{dl}$. Applying this to above eqn, with $\vec{dl}$ chosen to be a chunk of streamline $\vec{dr}$, we get: $dQ = 0$ along a streamline, where $Q \equiv \frac{q^2}{2} + \int \frac{dp}{\rho} + gz$.

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$$\therefore \; d\left(\frac{q^2}{2} + \int \frac{dp}{\rho} + gz\right) = 0 \quad \underline{\text{along a streamline}}.$$

$$\therefore \; \boxed{\frac{q^2}{2} + \int \frac{dp}{\rho} + gz = C}$$ where C is const along a streamline (might be dif on dif streamlines). This is Bernoulli's eq^{n} for steady, inviscid, barotropic flow.

If ρ is const then $\frac{1}{\rho}\nabla p = \nabla \frac{p}{\rho}$ and a derivation similar to that above yields,

$$\boxed{\frac{q^2}{2} + \frac{p}{\rho} + gz = C}$$ Bernoulli's eq^{n} for steady, inviscid const density flow. It's an energy eq^{n} (sum of kinetic energy, potential energy and "pressure potential" energy is const).

Case 2: If flow is irrotational ($\vec{\omega} = 0$ so $\vec{\omega} \times \vec{u} = 0$) and barotropic (and inviscid) can derive a different Bernoulli eq^{n}, which is valid whether or not flow is in a steady state. For irrotational, inviscid, barotropic flow (but possibly unsteady) Euler's eq^{ns} become:

$$\therefore \quad \frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{q^2}{2} + \int \frac{dp}{\rho} + gz\right) = 0$$

Since $\vec{\omega} = 0$ ($\nabla \times \vec{u} = 0$) there exists a velocity potential ϕ s.t.: $\vec{u} = \nabla\phi$ (can show that if $\vec{u} = \nabla\phi$ then $\nabla \times \vec{u} = 0$. Reverse is true as well but more difficult to show).

$$\therefore \quad \frac{\partial \vec{u}}{\partial t} = \frac{\partial}{\partial t}\nabla\phi = \nabla\frac{\partial \phi}{\partial t}$$

$\therefore$ Euler's eq^{ns} becomes:

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \int \frac{dp}{\rho} + gz\right) = 0 \qquad \text{(everywhere)}$$

If $\nabla(\;) = 0$ everywhere then ( ) is spatially const .

So get <u>Bernoulli's eq^{n}</u> for <u>inviscid, barotropic, irrot</u> flow:

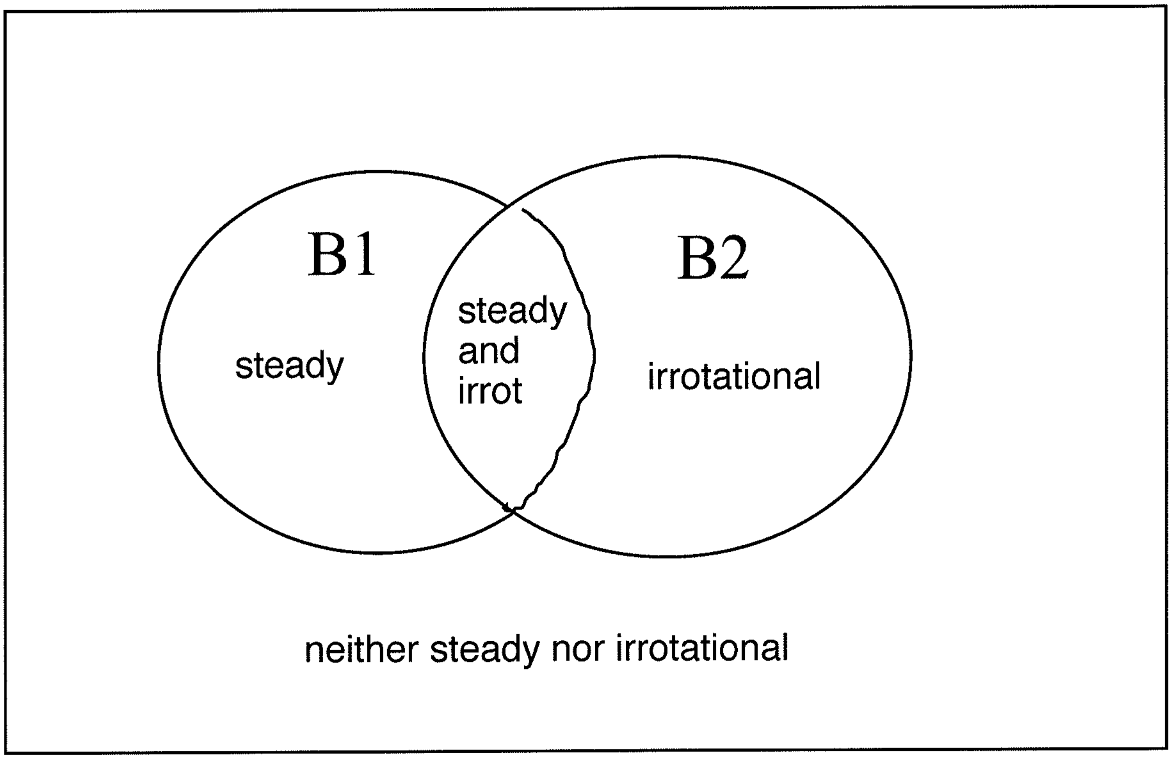
$$\boxed{\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \int \frac{dp}{\rho} + gz = C} \qquad \underline{\text{same const everywhere}}$$

[well, C can be a f^{n} of time but time dependence is irrelevant.]

If density is constant the appropriate B eqn for irrot flow is:

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \frac{p}{\rho} + gz = C} \qquad \underline{\text{same const everywhere}}$$

Let B1 be a version of Bernoulli's eqn appropriate for steady state flows, and B2 be a version of Bernoulli eqn appropriate for irrotational flows. Then we can draw a conceptual Venn diagram showing when you can use either, both or neither of these two Bernoulli eqns:



In this diagram we consider all possible flows that are inviscid and barotropic (or const density). For flows that are both steady and irrotational (intersection of B1 and B2) you can use either B1 or B2. In such a case, the use of B2 may be preferable to B1 since the constant "C" in B2 is a constant everywhere, not just along a streamline. If a flow is neither steady nor irrotational, you're out of luck -- can't use B1 or B2. However if the flow is "almost" steady (i.e., if the unsteady term is really tiny) then you might be able to get by with a steady-state approximation.