METR 5113, Advanced Atmospheric Dynamics I
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Monday, 17 Rocktober 2018 (lecture 24)

1 handout: problem set 4

<u>Barrel problem continued</u>

recall that for barrel prob: $\frac{V^2}{2} + gh = \frac{v^2}{2}$ (***)

get a more accurate relation than Torricelli's thm by keeping V
-- get V from mass cons eq^n, VA = va.

$\therefore \quad V = \frac{a}{A} v$

$\therefore$ (***) becomes:

$$v^2\left(\frac{a^2}{A^2} - 1\right) = -2gh$$

$$v^2 = \frac{2gh}{1 - a^2/A^2}$$

(****) $$\boxed{v = \sqrt{\frac{2gh}{1 - a^2/A^2}}}$$

When does barrel run dry? Happens at time T when h(T) = 0.
Let's find a formula for h(t) and then find T such that h(T) = 0.

Write (****) as eq^n involving h as the only dependent variable.

$$V = -\frac{dh}{dt}$$ (need minus sign since dh/dt is negative and we've been treating V -- speed -- as positive).

$$v = \frac{A}{a} V = -\frac{A}{a}\frac{dh}{dt}$$

So (****) becomes,

$$-\frac{A}{a}\frac{dh}{dt} = \sqrt{\frac{2gh}{1 - \frac{a^2}{A^2}}}$$ mult by -a/A

$$\frac{dh}{dt} = -\sqrt{h}\sqrt{\frac{2g}{A^2/a^2 - 1}}$$ (first order nonlinear ode)

separate variables:

$$\frac{dh}{\sqrt{h}} = -dt\sqrt{\frac{2g}{A^2/a^2 - 1}}$$ integrate it

$$2\sqrt{h} = -t\sqrt{\frac{2g}{A^2/a^2 - 1}} + D$$

Apply initial condition $h(0) = h_0$ in above result, get: $D = 2\sqrt{h_0}$

With D now known, can solve for h as:

$$\therefore\ h = \left[\sqrt{h_0} - t\sqrt{\frac{g}{2(A^2/a^2 - 1)}}\right]^2$$

Water runs out at the special time T, when $h(T) = 0$:

$$0 = \left[\sqrt{h_0} - T \sqrt{\frac{g}{2(A^2/a^2 - 1)}} \right]^2$$

$$T = \sqrt{\frac{2h_0(A^2/a^2 - 1)}{g}}$$

Time to run dry is longer for deeper barrels (h_0 ↑) and wider barrels (A ↑) ... and on planets with weaker gravity (g ↓).

2-D incompressible irrotational flows

This topic is covered in Ch7 of Kundu (6th ed) and Ch 6 of Kundu (editions 1-5)

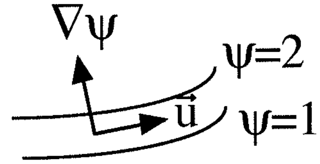
Suppose flow is 2-D [u = u(x, y), v = v(x, y), w = 0], and incompressible. Incomp cond^n for 2-D flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

∴ Can introduce a streamfunction ψ such that,

(A) $\boxed{u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}}$ (looked at this about a month ago)

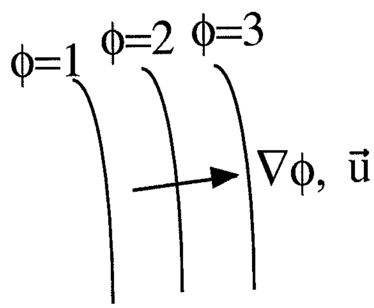
This means that $\vec{u} = -\hat{k} \times \nabla\psi$



Suppose flow is also irrotational: $\vec{\omega} = 0$

∴ $\vec{u} = \nabla\phi$ (ϕ is velocity potential)

(B) $$\boxed{u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}}$$



Flows that are incomp and irrot are known as potential flows.

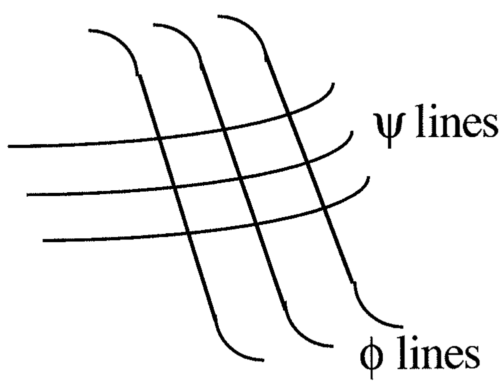
Comparing (A) with (B) we get:

$$\boxed{\begin{aligned} \frac{\partial\phi}{\partial x} &= \frac{\partial\psi}{\partial y}, \\ \frac{\partial\phi}{\partial y} &= -\frac{\partial\psi}{\partial x} \end{aligned}}$$

Cauchy-Riemann equations

If you know ϕ, you can solve C-R eq^{ns} for ψ (and vice-versa).

Using C-R eq^{ns} can show that lines of const ϕ are orthogonal ($\perp$) to lines of const ψ., i.e., $\nabla\phi \cdot \nabla\psi = 0$:



Proof:

$$\nabla\phi \cdot \nabla\psi = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}\right) \cdot \left(\frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j}\right)$$

$$= \boxed{\frac{\partial\phi}{\partial x}}\frac{\partial\psi}{\partial x} + \boxed{\frac{\partial\phi}{\partial y}}\frac{\partial\psi}{\partial y} \qquad \text{Use C-R eqns}$$

$$= \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} = 0$$

Use C-R eq^{ns} (2 eq^{ns} in 2 unknowns) to get 1 eq^{n} in 1 unknown:

Take $\frac{\partial}{\partial x}$ of 1st C-R eq^{n}: $\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)$

Take $\frac{\partial}{\partial y}$ of 2nd C-R eq^{n}: $\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right)$

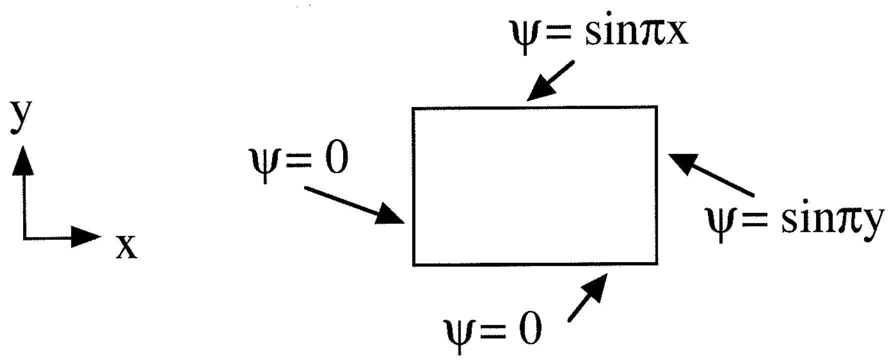
Add 'em up: $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x} = 0$

$\therefore$ $\boxed{\nabla^2\phi = 0}$ <u>Laplace's eq^{n}</u>. A 2nd order linear elliptic pde

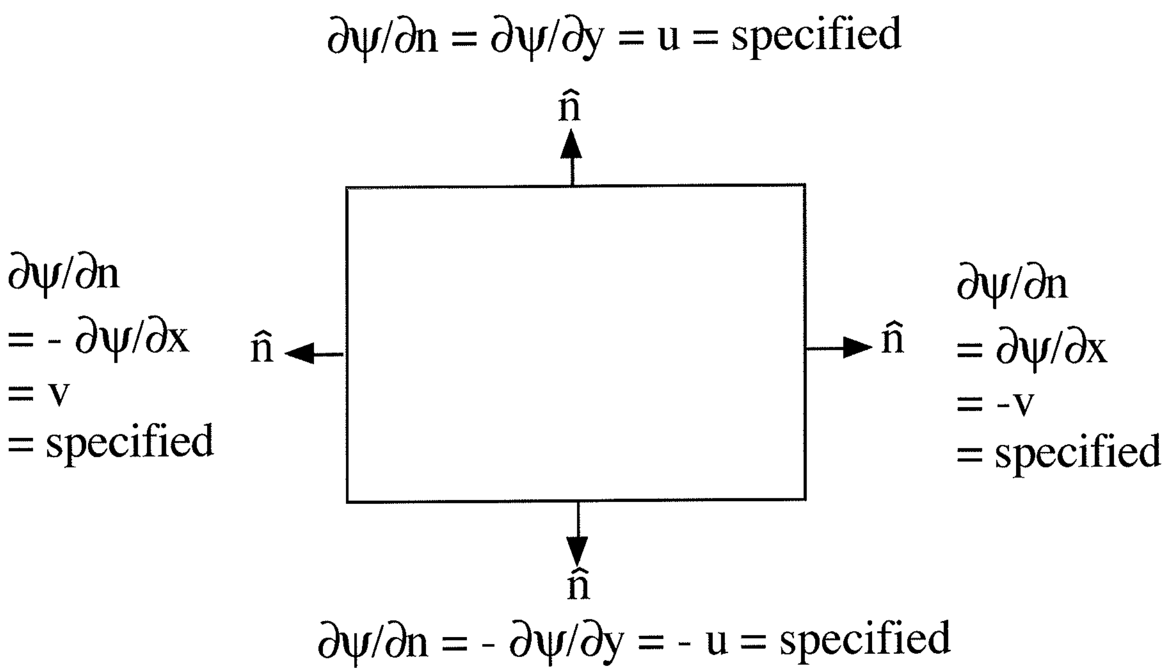
Similarly, can show $\boxed{\nabla^2\psi = 0}$. So both ϕ and ψ satisfy 2D Laplace eq^{ns}.

To solve Laplace's eq^{n} (e.g. $\nabla^2\psi = 0$) need one of 2 kinds of boundary conditions.

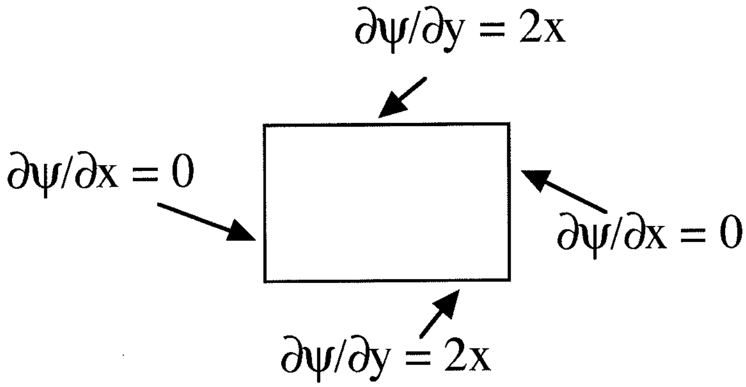
(1) <u>Dirichlet condⁿ</u>: Dependent variable is specified on bdry:



(2) <u>Neumann condⁿ</u>: Normal derivative of dep variable is specified on bdry. ($\partial\psi/\partial n$ is specified, where $\partial\psi/\partial n \equiv \hat{n}\cdot\nabla\psi$ and $\hat{n}$ is unit outward normal to bdry) i.e., velocity component <u>tangential to bdry</u> is specified



e.g.:



Can also "mix and match":