METR 5113, Advanced Atmospheric Dynamics I
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2D incompressible irrotational flow (cont$^{\underline{d}}$)

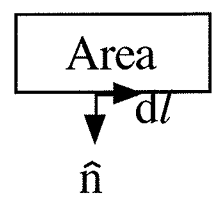
Careful w/ purely Neumann b.c.! There's a compatibility cond$^{\underline{n}}$ that boundary data must satisfy if problem is to have a sol^n.

Suppose $\nabla^2\psi = 0$ w/ Neumann b.c.: $\partial\psi/\partial n$ = specified on bdry. Integrate Lap eq^n over the flow domain.

$$\therefore \int \nabla^2\psi \, dA = 0$$

$$\therefore \int \nabla \cdot \nabla\psi \, dA = 0$$

Use 2D div thm: $\int \nabla \cdot \nabla\psi \, dA = \oint \nabla\psi \cdot \hat{n} \, dl$



$$\therefore \oint \nabla\psi \cdot \hat{n} \, dl = 0$$

Using def^n of normal deriv: $\partial\psi/\partial n = \hat{n} \cdot \nabla\psi$, we get

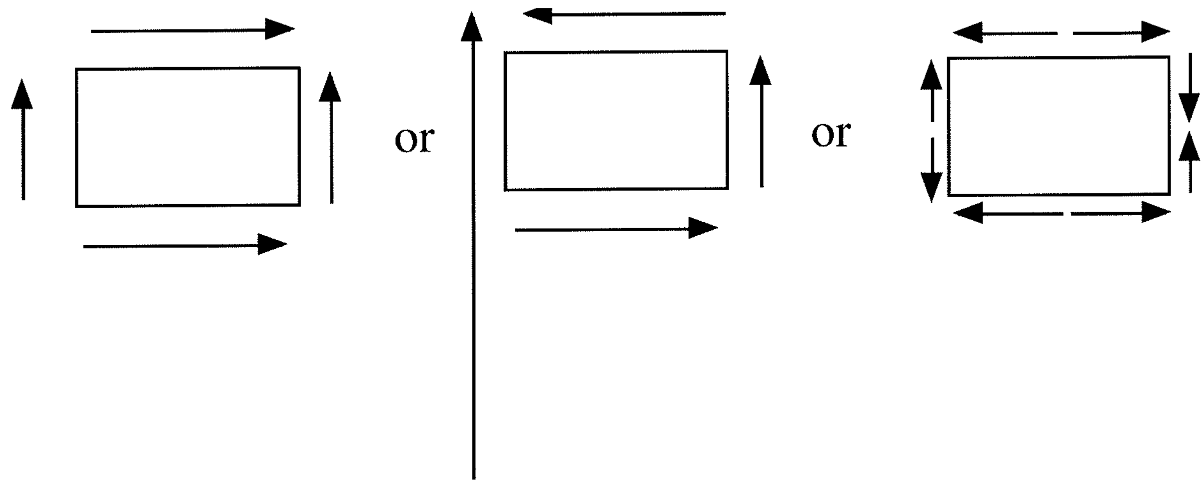
$$\therefore \boxed{\oint \frac{\partial\psi}{\partial n} \, dl = 0}$$

If you use purely Neumann b.c. to solve Lap eq^n, make sure compatibility cond^n is satisfied by $\partial\psi/\partial n$ data-- or the problem has no solution [integrated Lap eqn would contradict b.c.]

For $\nabla^2\psi = 0$ the compatibility cond^n means "integrated tangential wind on boundary is 0". This condition is consistent with Stokes th$^{\underline{m}}$: since flow is irrot, vort is 0 so area integral of vort is 0, so integrated tangential wind on bdry is 0.

For $\nabla^2\phi = 0$, compatibility cond^n is $\oint \partial\phi/\partial n \, dl = 0$, which means "integrated normal comp wind on bdry is 0". It agrees with <u>div th^m</u>: since flow is 2D incomp (horiz non-divergent), area integral of horiz divergence is 0, so integrated normal comp wind on bdry is 0.

Pictorial examples of flows on bdry satisfying $\oint \partial\psi/\partial n \, dl = 0$:



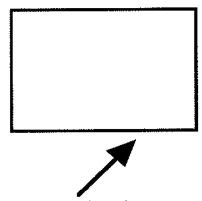
Careful with Neumann b.c. with Poisson eqn and other elliptic eq^{ns}. Doesn't mean Neumann b.c. is "bad" -- many times this is the one that you should impose -- but be careful. If you specify Dirichlet b.c. on bdry (even on part of it), there's nothing to worry about since $\oint \partial\psi/\partial n \, dl$ is not being imposed.

<u>Uniqueness Proof for Laplace's eq^n</u>

Suppose we find a function ψ_1 satisfying $\nabla^2\psi_1 = 0$ on flow domain with Dirichlet b.c. ψ_1 = specified on bdry. Is ψ_1 the only sol^n satisfying Laplace's eq^n w/ <u>same</u> Dirichlet b.c.?
Lets see what happens if we suppose there's another function ψ_2 satisfying <u>Laplace's eq^n</u> and <u>the same Dirichlet b.c</u>.

Define $\psi_d \equiv \psi_2 - \psi_1$, the difference btw the two sol^{ns}.

On bdry:



$\psi_1 = \psi_2$ on bdry

$\therefore\ \psi_2 - \psi_1 = 0$ on bdry $\quad \therefore\ \psi_d = 0$ on bdry

Within the flow domain:

$$\nabla^2\psi_d = \nabla^2\left(\psi_2 - \psi_1\right) = \nabla^2\psi_2 - \nabla^2\psi_1 = 0 - 0 = 0$$

$\therefore\ \psi_d$ satisfies Lap eq^n , and $\psi_d = 0$ on bdry.

$$\nabla^2\psi_d = 0$$

$$\therefore\ \psi_d\nabla^2\psi_d = 0$$

$$\therefore\ \int \psi_d\nabla^2\psi_d\, dA = 0$$

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scratch paper to help us integrate the above integral by parts:

$\nabla \cdot (\psi_d\nabla\psi_d) = \psi_d\nabla \cdot \nabla\psi_d + \nabla\psi_d \cdot \nabla\psi_d$ (vector product rule)

$$\therefore\quad \nabla \cdot (\psi_d\nabla\psi_d) = \psi_d\nabla^2\psi_d + \left|\nabla\psi_d\right|^2$$

$$\therefore\quad \psi_d\nabla^2\psi_d = \nabla \cdot (\psi_d\nabla\psi_d) - \left|\nabla\psi_d\right|^2$$

$$\therefore \int \left[\boxed{\nabla \cdot \left(\psi_d \nabla \psi_d\right)} - \left|\nabla \psi_d\right|^2 \right] dA = 0$$

use div th^m on first term (assuming 1st derivs are continuous)

$$\therefore \oint \psi_d \frac{\partial \psi_d}{\partial n}\, dl - \int \left|\nabla \psi_d\right|^2 dA = 0$$

bdry (line) integral

But $\psi_d = 0$ on bdry $\therefore$ line integral = 0!

$$\therefore \int \left|\nabla \psi_d\right|^2 dA = 0$$

$\left|\nabla \psi_d\right|^2$ must be ≥ 0. But if $\left|\nabla \psi_d\right|^2 > 0$ <u>anywhere</u> in domain then the integral would be positive, not 0. So $\left|\nabla \psi_d\right|^2$ must be 0 everywhere. So $\nabla \psi_d = 0$ everywhere.

$\therefore \quad \psi_d = \text{const}$ (everywhere)

But $\psi_d = 0$ on bdry $\therefore$ const = 0.

$\therefore \quad \psi_d = 0$ everywhere

$\therefore \quad \psi_2 - \psi_1 = 0$ everywhere

$\therefore \quad \psi_2 = \psi_1$ everywhere

$\therefore$ There really is only 1 solution. (i.e., sol^n is unique).

Note: Uniqueness proof relied on validity of div th^m. Proof breaks down for 2-D flow around infinite cylinder of any crosssection shape. In that case, sol^n of Lap eq^n w/ Dirichlet b.c. not unique in flow domain [nonuniqueness is associated with possibility that ψ is multivalued] See section 6.10 of Kundu.

What about solution uniqueness of Lap eqn w/ Neumann b.c.? Suppose we find a function ψ_1 satisfying $\nabla^2\psi_1 = 0$ everywhere, with $\partial\psi_1/\partial n$ specified on bdry [satisfying compatability cond^n]. Is ψ_1 the only sol^n satisfying Lap eq^n w/ <u>same</u> Neumann b.c.?

Suppose ψ_2 satisfies Laplace's eq^n w/ <u>same</u> <u>Neumann b.c.</u> as ψ_1.

Define $\psi_d \equiv \psi_2 - \psi_1$.

Retrace steps from before. Find that:

$$\nabla^2\psi_d = 0, \qquad \text{and} \quad \partial\psi_d/\partial n = 0 \text{ on bdry}$$

$$\therefore \int \psi_d \nabla^2 \psi_d \, dA = 0$$

$$\therefore \oint \psi_d \frac{\partial \psi_d}{\partial n} \, dl - \int |\nabla\psi_d|^2 dA = 0$$

(where we've assumed it's legal to use div th^m).

But $\partial\psi_d/\partial n = 0$ on bdry (in prev proof $\psi_d = 0$ on bdry)

$$\therefore \oint \psi_d \frac{\partial \psi_d}{\partial n} \, dl = 0$$

$$\therefore \int |\nabla\psi_d|^2 dA = 0$$

$\therefore \quad \nabla\psi_d = 0$ everywhere.

$\therefore \quad \psi_d = \text{const}$ (everywhere)

But b.c. $\partial\psi_d/\partial n = 0$ is satisfied for <u>any choice of const</u>.

$\therefore \quad \psi_2 - \psi_1 = \text{const}$ (everywhere)

$\therefore \quad \psi_2 = \psi_1 + \text{const}$ (everywhere)

For Neumann b.c., sol^n of Lap eq^n is "almost unique": <u>unique apart from an additive constant</u>. However, gradients of ψ <u>are</u> unique (if conditions of div th^m are met). So u, v soln is unique.

Non-uniqueness of <u>Poisson eqn</u> [e.g. $\nabla^2 p = F$ where p is pressure and F is a known function of u,v,w] with Neumann b.c. is an important problem for thermodynamic retrieval using u,v,w retrieved from dual-Doppler radar data -- need sounding data of pressure to resolve non-uniqueness problem in this case. [Try working through uniqueness/non-uniqueness result for Poisson eqn for Dirichlet b.c. case and Neumann b.c. case and see what you get.]