METR 5113, Advanced Atmospheric Dynamics I
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Monday, 22 Rocktober 2018 (lecture 26)

- 1 handout: Natural coords

How to solve Laplace's equation.

(1) Numerical methods

- direct solvers
- iterative solvers, e.g., successive over-relaxation (SOR)

(2) Analytical methods

- separation of variables
- Green's function method.
- Complex variable theory.

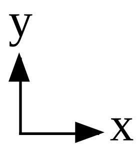
However, can sometimes get a simple qualitative picture of the solution without all the math by considering natural coordinates. Lets look at natural coordinates and then put them to use.

- Go through overhead transparency/handout on natural coords.

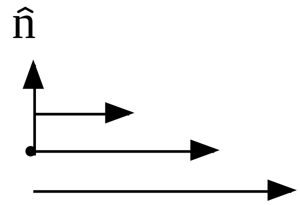
A positive value of vorticity in natural coordinates means vorticity vector pointing out of page.

Vertical vorticity in natural coordinates: $\zeta = -\frac{\partial V}{\partial n} + \frac{V}{R}$

Consider unidirectional shear flow in xy plane. Since $|R| = \infty$ in this case, vort is associated only with shear (curvature term is 0):



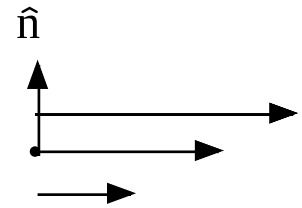
z out of page
$\hat{k}$ " " "



$\frac{\partial V}{\partial n} < 0$

$\therefore\ \zeta > 0$

so vort vector $\vec{\omega} = \zeta\hat{k}$
points out of pg ($\hat{k}$ dir^n)



$\frac{\partial V}{\partial n} > 0$

$\therefore\ \zeta < 0$

so vort vector $\vec{\omega} = \zeta\hat{k}$
points into page ($-\hat{k}$ dir^n)

Scratch paper to show that if a < b then -a > -b, i.e., multiplication by a negative switches sign of inequality.
It makes sense graphically. Suppose a < b:

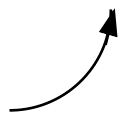
_____._____._____.__________
0 a b

then: _____._____._____.__________
-b -a 0

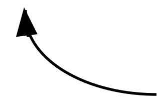
so - a > -b (-a is greater in value than -b)

Can summarize these results in a <u>right hand rule for shear vort</u>: To find dir^n of the vorticity at a point of interest, align fingers of your right hand with velocity vector at that point, then curl your fingers in the dir^n of the <u>slower</u> flow. Your thumb indicates dir^n of vorticity. [If such an action isn't natural and would hurt your fingers then flip your hand over and then curl your fingers].

Curvature vorticity at a point of interest only depends on V and the curvature of the trajectory through that point.



R > 0



R < 0

curvature vort = V/R > 0
V (speed) is always pos,
so vort vector associated
w/ curv vort is out of page
(so in $\hat{k}$ dir^n.)

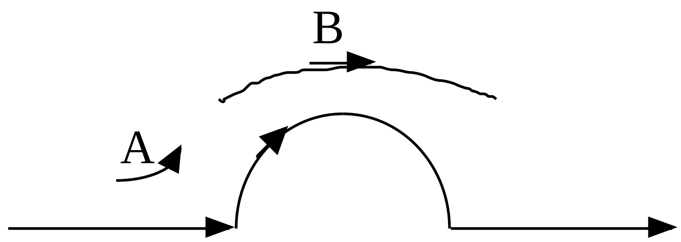
V/R < 0
So vort vector associated
w/ curv vort is into page
(so $-\hat{k}$ dir^n.)

Can summarize this result in <u>right hand rule for curvature vort</u>: To find the dir^n of vorticity at a point of interest, curl the fingers of your right hand along the trajectory at that point (in direction of flow). Your thumb indicates the dir^n of vort. [If this action isn't natural, flip your hand over and then curl your fingers].

Can use natural coords to describe qualitative behavior of 2-D incomp, irrot flows (satisfies Laplace's eqn for ψ). Vorticity is 0, but think of zero vort as sum of a shear vort and an equal-but-opposite curvature vort: $-\frac{\partial V}{\partial n} + \frac{V}{R} = 0$. Revisit some examples:

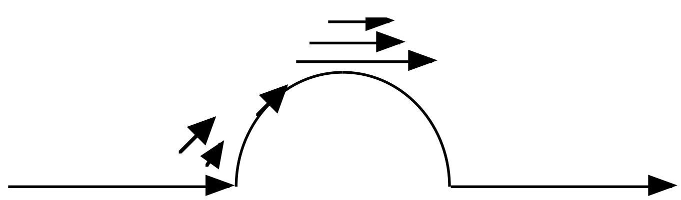
e.g., Uniform flow over a cylinder

In inviscid flow, impermeability cond^n is the only b.c. that can be imposed on solid bdry (can't impose no-slip): no flow normal to bdry. So velocity is tangent to bdry. So bdry is a streamline. Draw this bdry streamline (can call it the "central" streamline since the flow is symmetric about it). This streamline suggests the sense of the curvature in the flow above it.



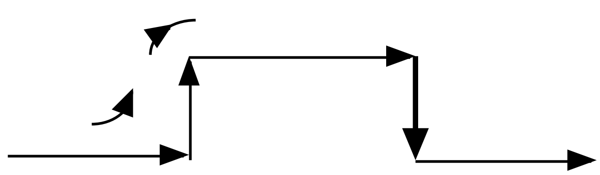
At point A curvature vort is out of page so shear vort must be into page. Right hand rule for shear vort says speed must be decreasing as lower corner point is approached.

At point B curvature vort is into page so shear vort must be out of page. So speed must be increasing as top of cylinder is approached from above. So:



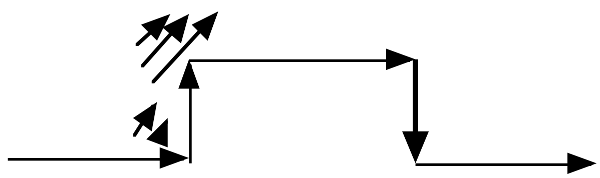
e.g., Uniform flow over a Walmart

Again, imp cond^{n} reveals that lower bdry must be a streamline. Curvature in flow above the lower boundary is as indicated:



So, as in cylinder case, we have out-of-page curvature vort in lower corner. So have shear vort pointing into page. So speed decreases as lower corner point is approached (this is consistent with the notion that the lower corner point is a stagnation point).

Near top corner, have curvature vort pointing into page. So have shear vort pointing out of page. So speed increases as corner point is approached. But as corner is approached, radius of curvature approaches 0 so V/R becomes infinite. So shear vort become infinite. Speed increases as corner point is approached and becomes infinite at corner! A singularity. [Note: in viscous flow, friction prevents singularity from occuring, but still get very fast flow at corner.]



Why didn't similar logic lead to a singularity at the lower corner? At the lower corner the radius of curvature is 0 but it approaches 0 through positive values (whereas at top corner it approaches 0 through negative values). So the shear vorticity is such that the speed decreases as the lower corner point is approached. But cannot have negative speeds; so as R goes to 0, V must also go to zero.