METR 5113, Advanced Atmospheric Dynamics I
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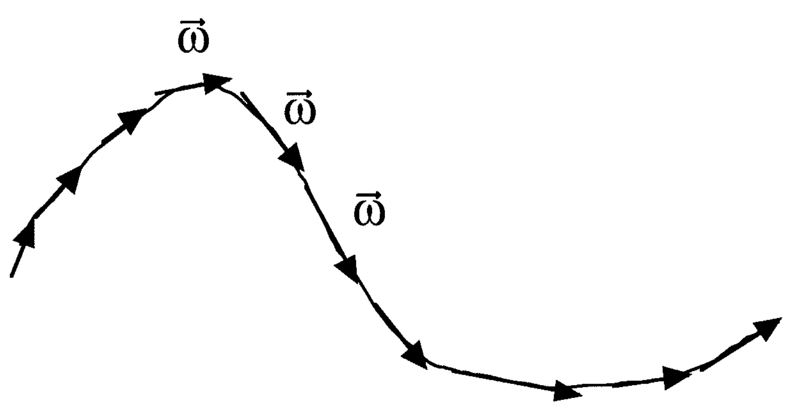
Wednesday, 24 Rocktober 2018 (lecture 27)

Kinematics of Vorticity

vorticity $\vec{\omega} \equiv \nabla \times \vec{u}$

A line that is everywhere tangential to the local vorticity vector is a vortex line (or vortex filament).



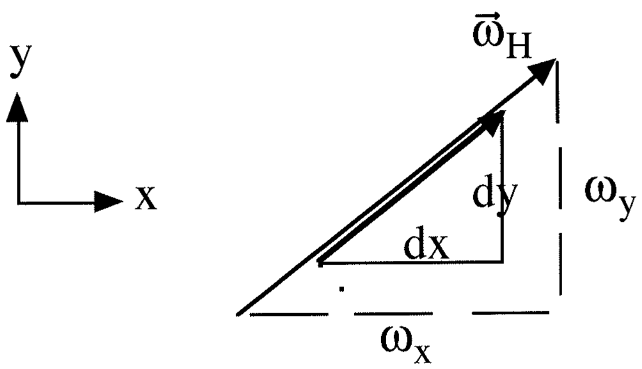
Note: A vortex line is not the same thing as a line vortex ["line vortex" is a vr-vortex or name given to a concentrated vortex -- vorticity concentrated in a small tube-like volume]

ODEs for vortex lines are analogous to ODEs for streamlines. Derive them analytically by expanding out

$$\vec{dx} \times \vec{\omega} = 0\,, \quad \text{[for streamlines we expanded out } \vec{dx} \times \vec{u} = 0\,]$$

where $\vec{dx} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ is a chunk of the vortex line.

Or can derive them geometrically by considering the diagram:



where $\vec{\omega}_H \equiv \omega_x \hat{i} + \omega_y \hat{j}$

so $\dfrac{dy}{dx} = \dfrac{\omega_y}{\omega_x}$, similarly: $\dfrac{dz}{dx} = \dfrac{\omega_z}{\omega_x}$. Can put them in the form:

$$\boxed{\frac{dy}{\omega_y} = \frac{dx}{\omega_x} = \frac{dz}{\omega_z}}$$

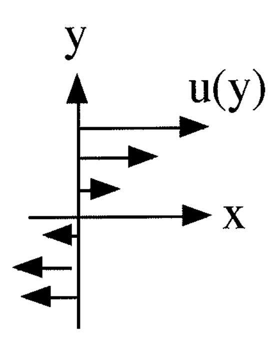
or can introduce a parameter "s" such that $\dfrac{dy}{\omega_y} = \dfrac{dx}{\omega_x} = \dfrac{dz}{\omega_z} = ds$,

from which follows parametric form of odes for the vortex line:

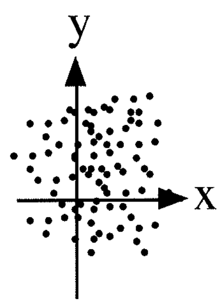
$$\boxed{\frac{dx}{ds} = \omega_x, \quad \frac{dy}{ds} = \omega_y, \quad \frac{dz}{ds} = \omega_z}$$

Vortex lines are present whenever there is vorticity in flow (don't need to have a vortex).

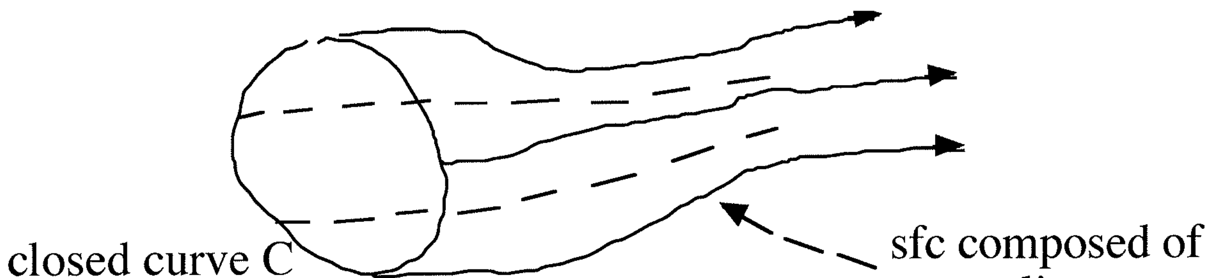
e.g. shear flow $u = u(y)$, $v = 0$, $w = 0$



In this case vorticity is all in vertical dir^n, $\vec{\omega} = -\frac{\partial u}{\partial y}\hat{k}$ (in $-\hat{k}$ dir^n or $+\hat{k}$ dir^n) so vortex lines are all in vertical dir^n.

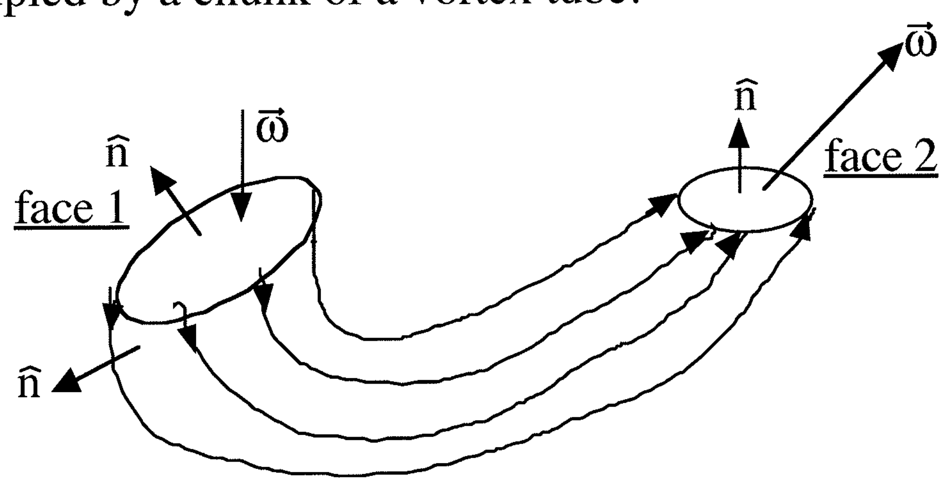


Vortex lines passing through any closed curve form a tubular sfc known as a vortex tube [analogous to construction of streamtube from streamlines passing through a closed curve]



$\vec{\omega} \cdot \hat{n} = 0$ on this sfc
($\hat{n}$ is unit outward normal)

Recall $\nabla \cdot \vec{\omega} = 0$ for all flows (done in prob set) $\therefore \int \nabla \cdot \vec{\omega}\, dV = 0$ for any fluid volume. In particular, consider the volume occupied by a chunk of a vortex tube.



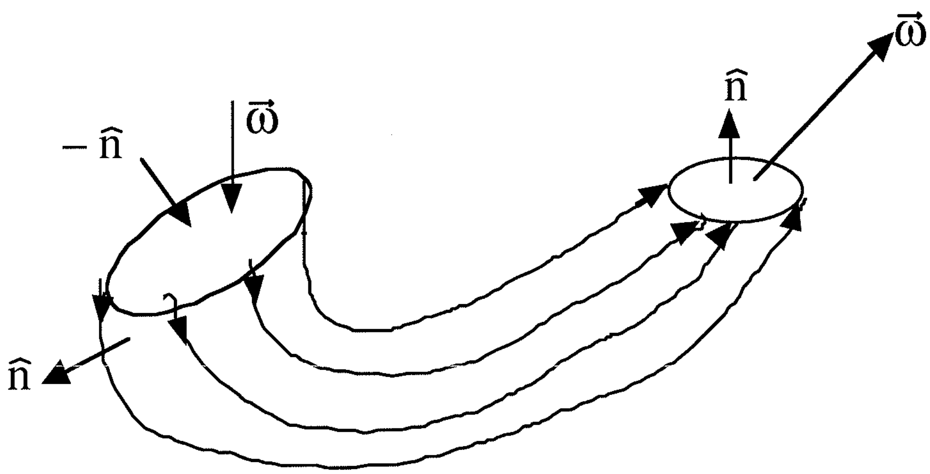
But div thm says $\int \nabla \cdot \vec{\omega}\, dV = \int \vec{\omega} \cdot \hat{n}\, dA$. Since l.h.s. is 0, the r.h.s must be 0. So $\int \vec{\omega} \cdot \hat{n}\, dA = 0$. This area integral is over closed surface bounding chunk of tube (sides + two end faces).

0 on sides of tube

$$\therefore \int_{\text{face 1}} \vec{\omega} \cdot \hat{n}\, dA + \int_{\text{face 2}} \vec{\omega} \cdot \hat{n}\, dA + \int_{\text{sides}} \boxed{\vec{\omega} \cdot \hat{n}}\, dA = 0$$

$$\therefore -\int_{\text{face 1}} \vec{\omega} \cdot \hat{n}\, dA = \int_{\text{face 2}} \vec{\omega} \cdot \hat{n}\, dA$$

$$\therefore \int_{\text{face 1}} \vec{\omega} \cdot (-\hat{n})\, dA = \int_{\text{face 2}} \vec{\omega} \cdot \hat{n}\, dA$$

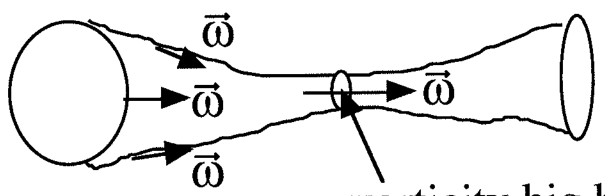


[now use fact that "average = integral divided by interval"]

$$\therefore \quad A_1 \boxed{\frac{1}{A_1} \int \vec{\omega} \cdot (-\hat{n})\, dA} = A_2 \boxed{\frac{1}{A_2} \int \vec{\omega} \cdot \hat{n}\, dA}$$

average inward comp of vorticity on face 1 — average outward comp of vorticity on face 2

More generally, $A\varpi$ has <u>same value</u> on any x-section through a given vortex tube. So $\boxed{A\varpi = \text{const}}$ for a vortex tube, where A is cross-section area through tube and ϖ is ave vorticity comp $\perp$ to cross-section area. $A\varpi$ is "strength" of vortex tube. So vortex tube strength is constant -- indep of where cross-section is taken.



Rewrite $A\varpi = \text{const}$ using Stokes Th^m, get: $\oint \vec{u} \cdot \vec{dl} = \text{const}$. Here the curve is the intersection of the tube and cross-section area. So circulation is the same for any cross-section along vortex tube.

These results are <u>kinematic</u> relations valid at a <u>fixed time</u>. How does vortex tube strength (or circulation) change with time? Need to consider <u>dynamics</u>.

<u>**Vorticity and Circulation Dynamics** (Ch5 Kundu 1st-6th ed)</u>

<u>Kelvin's Circulation Th^m</u>

Assume flow is:

(1) inviscid

(2) subject to conservative body forces (force can be expressed as gradient of something -- as in gravity force)

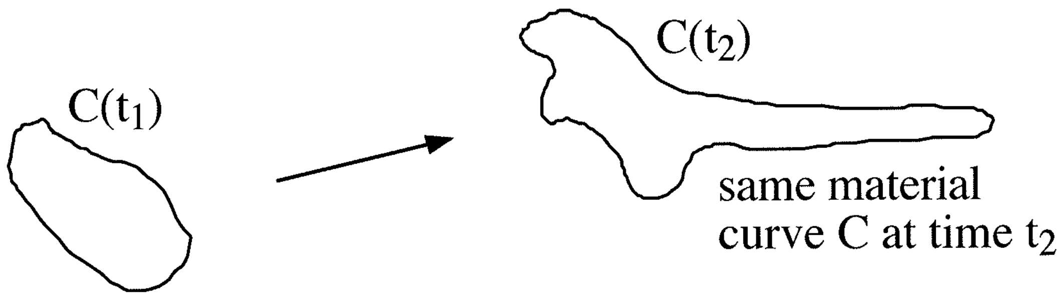
(3) barotropic, $\rho = \rho(p)$

(4) described in an inertial (non-accelerating) ref frame

Appropriate eq^{ns} of motion are Euler eq^{ns}:

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{g}, \qquad \text{where } \vec{g} = -\nabla gz$$

Consider an arbitrary closed curve C (arbitrary at a particular time, but once you've defined it, you're stuck with it). Curve is composed of air parcels. Follow curve as it moves around. C is always composed of same parcels -- same material -- it's a material curve.



Let $\Gamma(t) \equiv \oint \vec{u} \cdot \vec{dl}$ be circulation around C.

To see how Γ changes with time, derive a formula for $\frac{D\Gamma}{Dt}$, rate of change of circulation around closed material curve.