METR 5113, Advanced Atmospheric Dynamics I
Alan Shapiro, Instructor
Friday, 26 Rocktober 2018 (lecture 28)

Kelvin's Circulation Th^m (continued)

Let $\Gamma(t) \equiv \oint \vec{u} \cdot \vec{dl}$ be circulation around closed material curve C.

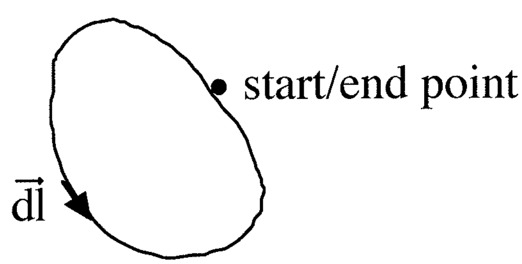
To see how Γ changes with time, derive a formula for $\frac{D\Gamma}{Dt}$.

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint \vec{u} \cdot \vec{dl} = \oint \boxed{\frac{D\vec{u}}{Dt}} \cdot \vec{dl} + \oint \vec{u} \cdot \frac{D}{Dt}\vec{dl}$$

substitute in from Euler's eqn

$$(*) \quad \frac{D\Gamma}{Dt} = \oint \left(-\frac{1}{\rho}\nabla p + \vec{g}\right) \cdot \vec{dl} + \oint \vec{u} \cdot \frac{D}{Dt}\vec{dl}$$

Want to put as many of the integrands as possible in the form of perfect differentials $d\chi$ since $\oint d\chi = \chi_{end} - \chi_{start} = 0$ (since start and end points are the same).



Look at pressure term in (*):

$$-\oint \frac{1}{\rho}\nabla p \cdot \vec{dl} = -\oint \frac{dp}{\rho}$$ where dp is change in p across $\vec{dl}$.

$$= -\oint \boxed{\frac{1}{\rho}\frac{dp}{d\rho}}\, d\rho \quad \text{(since flow is barotropic)}$$

a function of ρ. Call it $dF/d\rho$.

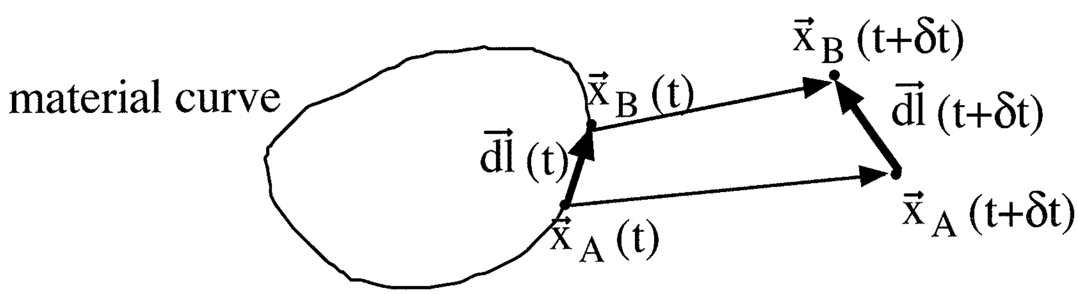
$$= -\oint \frac{dF}{d\rho}\, d\rho = -\oint dF = -(F_{end} - F_{start}) = 0$$

Look at gravity term in (*):

$$\oint \vec{g} \cdot \vec{dl} = -\oint \nabla \boxed{\Pi} \cdot \vec{dl} = -\oint d\Pi = 0$$

$\Pi \equiv gz$ is potential energy function

Now look at $\oint \vec{u} \cdot \frac{D}{Dt}\vec{dl}$ term in (*):



$\vec{dl}(t) \equiv \vec{x}_B(t) - \vec{x}_A(t)$ is element of curve at time t.

$\vec{dl}(t+\delta t) \equiv \vec{x}_B(t+\delta t) - \vec{x}_A(t+\delta t)$ is <u>same</u> element at time $t + \delta t$.

$$\frac{D}{Dt}\vec{dl} = \lim_{\delta t \to 0} \frac{\vec{dl}(t+\delta t) - \vec{dl}(t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{[\vec{x}_B(t+\delta t) - \vec{x}_A(t+\delta t)] - [\vec{x}_B(t) - \vec{x}_A(t)]}{\delta t}$$

Now use the Taylor expansions (in time):

$$\vec{x}_B(t+\delta t) = \vec{x}_B(t) + \vec{u}_B\delta t + \text{h.o.t.}$$

$$\vec{x}_A(t+\delta t) = \vec{x}_A(t) + \vec{u}_A\delta t + \text{h.o.t.}$$

these h.o.t involve terms of order $(\delta t)^2$ and higher

$$\therefore \frac{D}{Dt}\vec{dl} = \lim_{\delta t \to 0} \frac{\vec{x}_B(t) + \vec{u}_B\delta t - \vec{x}_A(t) - \vec{u}_A\delta t - [\vec{x}_B(t) - \vec{x}_A(t)]}{\delta t}$$

(h.o.t. vanish in limit of vanishing δt)

$$= \boxed{\vec{u}_B} - \vec{u}_A$$ do Taylor exp for $\vec{u}_B$ in space (neglect h.o.t.):

$$= \boxed{\vec{u}_A + \frac{\partial \vec{u}}{\partial x_i}dx_i} - \vec{u}_A = \frac{\partial \vec{u}}{\partial x_i}dx_i$$ [rewrite using $\vec{dl} = \hat{e}_i dx_i$]

$$= (\vec{dl}\cdot\nabla)\vec{u} = d\vec{u}$$ change in $\vec{u}$ from $\vec{x}_A$ to $\vec{x}_B$.

$$\therefore \frac{D}{Dt}\vec{dl} = d\vec{u}$$

$$\therefore \vec{u}\cdot\frac{D}{Dt}\vec{dl} = \vec{u}\cdot d\vec{u} = d\left(\frac{1}{2}\vec{u}\cdot\vec{u}\right)$$

$$\therefore \oint \vec{u}\cdot\frac{D}{Dt}\vec{dl} = \oint d\left(\frac{1}{2}\vec{u}\cdot\vec{u}\right) = 0$$

So all terms on rhs of (*) are 0!

We're left with <u>Kelvin's Circulation Theorem:</u>

$$\boxed{\frac{D\Gamma}{Dt} = 0}$$ for a material curve.

or

$$\boxed{\Gamma = \text{const}} \text{ for a material curve}$$

"absolute circulation is conserved following the motion of a material curve". Valid for barotropic, inviscid, flow subject to conservative body forces and described in an <u>inertial</u> ref frame.

Derivation made use of quantities on curve itself. As long as conditions for Kelvin's th^m hold on curve, the th^m is valid (doesn't matter if conditions are violated within curve).

Rewrite Kelvin's th^m using Stokes th^m, $\oint \vec{u} \cdot \vec{dl} = \int \vec{\omega} \cdot \hat{n}\, dA$.

So $$\boxed{\int \vec{\omega} \cdot \hat{n}\, dA = \text{const}}$$ (following the curve)

or:

$$\boxed{\overline{\omega} A = \text{const}}$$ (following the curve),

where $\overline{\omega}$ is the average normal component of the vorticity on area bounded by the curve.

<u>Example</u>: Consider a rotating vertical column of air and a material curve that is horizontal and encloses the column. If the curve remains horizontal but shrinks in size (so A decreases) then Kelvin's theorem says the average vertical-comp vorticity increases (the normal to the area enclosed by the horizontal curve points vertically).

<u>Example</u>: Consider, at t = 0, a rotating vertical column of air and a material curve that is horizontal and encloses the rotating

column. Suppose the curve doesn't shrink or expand but it changes its orientation -- say, it moves into a vertical orientation. Then the average normal-comp vorticity stays the same. But the "normal" direction has changed. In this example there is a rearrangment of vorticity: originally we had vertical component of vorticity, but end up with a horizontal component of vorticity.

Example: "Persistence of irrotational motion". Suppose conditions for validity of Kelvin's Th^m are satisfied (everywhere) and vorticity is initially 0. Consider a closed material curve. At t=0 the vorticity is 0 everywhere so it's 0 within area enclosed by this curve so integrated vorticity over that area is 0 $\therefore$ from Stokes th^m, the circ around curve is 0: $\Gamma(0) = 0$. Follow curve as it moves with the flow. Kelvin's th^m says $\Gamma(t) = \text{const}$. Since $\Gamma(0) = 0$, the const must be 0. So $\Gamma(t) = 0$. Get this result for all possible material curves. So $\Gamma(t) = 0$ for all time and for all material curves. But Stokes th^m says: $\Gamma = \int \vec{\omega} \cdot \hat{n}\, dA$

$\therefore \int \vec{\omega} \cdot \hat{n}\, dA = 0$ for all time and all material curves. So $\vec{\omega} \cdot \hat{n} = 0$ everywhere, for all time. And since this holds true for all possible $\hat{n}$ (choose curves with any orientation) $\vec{\omega} = 0$ everywhere, for all time.